Symplectic Rational Blow-Up and Embeddings of Rational Homology Balls

Dissertation Advisor: Professor Peter Kronheimer Tatyana Khodorovskiy

Symplectic Rational Blow-Up and Embeddings of Rational Homology Balls

Abstract

We define the symplectic rational blow-up operation, for a family of rational homology balls B_n, which appeared in Fintushel and Stern's rational blow-down construction [FS2]. We do this by exhibiting a symplectic structure on a rational homology ball B_n as a standard symplectic neighborhood of a certain 2-dimensional Lagrangian cell complex. We also study the obstructions to symplectically rationally blowing up a symplectic 4-manifold, i.e. the obstructions to symplectically embedding the rational homology balls B_n into a symplectic 4-manifold. First, we present a couple of results which illustrate the relative ease with which these rational homology balls can be smoothly embedded into a smooth 4-manifold. Second, we prove a theorem and give additional examples which suggest that in order to symplectically embed the rational homology balls B_n, for high n, a symplectic 4-manifold must at least have a high enough c_1^2 as well.

Contents

Acknowledgements

First and foremost, I would like to thank my advisor Professor Peter Kronheimer, for his incredible insight, guidance, patience and unwavering optimism. None of this work would have been possible without his direction and encouragement. I would also like to thank Professors Cliff Taubes and Tomasz Mrowka who, in addition to forming my thesis committee, have provided me with invaluable insights contributing to this work.

I am greatly indebted to my undergraduate advisor Professor Sylvain Cappell for supporting my pursuit of mathematics and getting me interested in topology from my freshman year.

I would also like to thank my friends and family for their unconditional love and support: my wonderful newlywed husband Leon Khodorovskiy, my mother Eugenia Kobylyatskaya for insisting that I always pursue what I am passionate about, my father Alexander Kobylyatskiy for buying me my first calculus book (which got all this started), my great friends Ana Caraiani, Steven Sivek and Regina Lazarovich, and last but not least my cat Kotya, who has always loyally and systematically slept on top of all my math books, notes and papers.

1. Introduction

In 1997, Fintushel and Stern [FS2] defined the rational blow-down operation for smooth 4-manifolds, a generalization of the standard blow-down operation. For smooth 4-manifolds, the standard blow-down is performed by removing a neighborhood of a sphere with self-intersection (-1) and replacing it with a standard 4-ball B^4. The rational blow-down involves replacing a negative definite plumbing 4-manifold with a rational homology ball. In order to define it, we first begin with a description of the negative definite plumbing 4-manifold C_n, $n \geq 2$, as seen in Figure 1, where each dot represents a sphere, S_i, in the plumbing configuration. The integers above the dots are the self-intersection numbers of the plumbed spheres: $[S_1]^2 = -(n+2)$ and $[S_i]^2 = -2$ for $2 \leq i \leq n-1$.

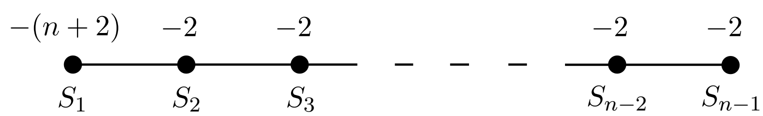


Figure 1. **Plumbing diagram of C_n, $n \geq 2$**

The boundary of C_n is the lens space $L(n^2, n-1)$, thus $\pi_1(\partial C_n) \cong H_1(\partial C_n; \mathbb{Z}) \cong \mathbb{Z}/n^2\mathbb{Z}$. (Note, when we write the lens space $L(p,q)$, we mean it is the 3-manifold obtained by performing $-\frac{p}{q}$ surgery on the unknot.) This follows from the fact that $[-n-2, -2, \ldots -2]$, with $(n-2)$ many (-2)'s is the continued fraction expansion of $\frac{n^2}{1-n}$.

Let B_n be the 4-manifold as defined by the Kirby diagram in Figure 2 (for a more extensive description of B_n, see section 3.1). The manifold B_n is a rational homology ball, i.e. $H_*(B_n; \mathbb{Q}) \cong H_*(B^4; \mathbb{Q})$. The boundary of B_n is also the lens space $L(n^2, n-1)$ [CH]. Moreover, any self-diffeomorphism of ∂B_n extends to B_n [FS2]. Now, we can define the rational blow-down of a 4-manifold X:

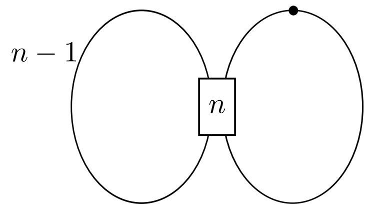


FIGURE 2. **Kirby diagram of B_n**

Definition 1.1. ([FS2], also see [GS]) Let X be a smooth 4-manifold. Assume that C_n embeds in X, so that $X = C_n \cup_{L(n^2,n-1)} X_0$. The 4-manifold $X_{(n)} = B_n \cup_{L(n^2,n-1)} X_0$ is by definition the *rational blow-down* of X along the given copy of C_n.

Fintushel and Stern [FS2] also showed how to compute Seiberg-Witten and Donaldson invariants of $X_{(n)}$ from the respective invariants of X. In addition, they showed that certain smooth logarithmic transforms can be alternatively expressed as a series of blow-ups and rational blow-downs. In 1998, (Margaret) Symington [Sy1] proved that the rational blow-down operation can be performed in the symplectic category. More precisely, she showed that if in a symplectic 4-manifold (M,ω) there is a symplectic embedding of a configuration C_n of symplectic spheres, then there exists a symplectic model for B_n such that the *rational blow-down* of (M,ω), along C_n is also a symplectic 4-manifold. (Note, we will often abuse notation and write C_n both for the actual plumbing 4-manifold and the plumbing configuration of spheres in that 4-manifold.)

As a result, Symington described when a symplectic 4-manifold can be symplectically rationally blown down. We would like to investigate the following question: **when can a symplectic 4-manifold be symplectically rationally blown up?** By *rational blow-up*, (at least in the smooth category) we mean the inverse operation of *rational blow-down*: if a 4-manifold has an embedded rational homology ball B_n, then we can rationally blow it up by replacing the B_n with the negative definite plumbing C_n. In order to do that, we first need to verify that rationally blowing up makes sense in the symplectic category. This is done in section 3.

The first step towards such a definition is to equip B_n with a symplectic structure, such that it is the "standard" symplectic neighborhood of a certain (2-dimensional) "Lagrangian core" $\mathcal{L}_{n,1}$ (see section 3.4.1 and for an illustration with $n = 3$ see Figure 9). For $n = 2$, $\mathcal{L}_{2,1}$ is simply a Lagrangian $\mathbb{R}P^2$. For $n \geq 3$, $\mathcal{L}_{n,1}$ is a cell complex consisting of an embedded S^1 and a 2-cell D^2, whose boundary "wraps" n times around the embedded S^1 (the interior of the 2-cell D^2 is an embedding). Furthermore, the cell complex $\mathcal{L}_{n,1}$ is embedded in such a way that the 2-cell D^2 is Lagrangian. We show, by mirroring the Weinstein Lagrangian embedding theorem, that a symplectic neighborhood of such an $\mathcal{L}_{n,1}$ is entirely standard. As a result, we show that we can obtain a symplectic model for B_n as a standard symplectic neighborhood of this Lagrangian complex $\mathcal{L}_{n,1}$.

Consequently, we prove that a symplectic 4-manifold (X, ω) can be symplectically rationally blown up provided there exists this "Lagrangian core" $\mathcal{L}_{n,1} \subset (X, \omega)$:

Theorem A. *(Theorem 3.8) Suppose we can find a "Lagrangian core" $\mathcal{L}_{n,1} \subset (X, \omega)$, (as in Definition 3.7), then for some small $\lambda > 0$, there exists a symplectic embedding of $(B_n, \lambda\omega_n)$ in (X, ω), and for some $\lambda_0 < \lambda$ and $\mu > 0$, there exists a symplectic 4-manifold (X', ω') such that $(X', \omega') = ((X, \omega) - (B_n, \lambda_0\omega_n)) \cup_\phi (C_n, \mu\omega'_n)$, where ϕ is a symplectic map, and (B_n, ω_n) and (C_n, ω'_n) are the symplectic manifolds as defined in section 3.3. (X', ω') is called the* ***symplectic rational blow-up*** *of (X, ω).*

In Theorem A above, the scaling coefficient λ, regulates the "size" of the rational homology ball B_n that is removed from the symplectic manifold (X, ω), just like in the definition of the regular symplectic blow-up operation, where one chooses the size of the 4-ball being removed. The scaling coefficient μ regulates the symplectic volume of C_n which can "fit back into" in place of the removed symplectic volume of B_n.

After defining the symplectic rational blow-up operation, we proceed to investigate the following question: **what are the obstructions to symplectically embedding the rational homology balls B_n into a symplectic 4-manifold?** First, in section 4, we tackle this question for smooth embeddings, in an attempt to determine whether the obstructions to symplectically embedding the B_n occurs on the smooth level. We prove the following results regarding smooth embeddings of the rational homology balls B_n:

Theorem B. *(Theorem 4.1) Let V_{-n-1} be a neighborhood of a sphere with self-intersection number $(-n-1)$. There exists an embedding of the rational homology balls $B_n \hookrightarrow V_{-n-1}$, for all $n \geq 2$.*

Theorem C. *(Theorem 4.2) Let V_{-4} be a neighborhood of a sphere with self-intersection number (-4). For all $n \geq 3$ odd, there exists an embedding of the rational homology balls $B_n \hookrightarrow V_{-4}$. For all $n \geq 2$ even, there exists an embedding of the rational homology balls $B_n \hookrightarrow B_2 \# \overline{\mathbb{C}P^2}$.*

Theorems B and C above show that there is little obstruction to *smoothly* embedding the rational homology balls B_n into a smooth 4-manifold. In particular, Theorem C implies that if a smooth 4-manifold X contains a sphere with self-intersection (-4), then one can smoothly embed the rational homology balls B_n into X for all odd $n \geq 3$.

One of the implications of Theorem C is that for a given smooth 4-manifold X, there does not exist an N, such that for all $n \geq N$ one cannot find a smooth embedding $B_n \hookrightarrow X$. In the setting of this sort in algebraic geometry, for rational homology ball smoothings of certain surface singularities, such a bound on n does exist, in terms of (c_1^2, χ_h) invariants of an algebraic surface [KSB, Wa]. Therefore, for the case of symplectic embeddings of the rational homology balls B_n, if we model our symplectic manifold such that it resembles a surface of general type, we can make the following conjecture:

Conjecture 1.2. *Let (X,ω) be a symplectic 4-manifold, such that:*

- $b_2^+(X) > 1$ *and*
- $[c_1(X,\omega)] = -[\omega]$ *as cohomology classes,*

then there exists an N, such that for all $n \geq N$ there does not exist a symplectic embedding $B_n \hookrightarrow (X,\omega)$.

The condition $[c_1(X,\omega)] = -[\omega]$, implies that (X,ω) does not contain any spheres of self-intersection (-1) or (-2) and $c_1^2(X,\omega) \geq 1$, resembling a surface of general type with an ample canonical divisor.

In section 5, we show a result that is a first step in proving the above conjecture. We observe that if we impose the condition $n \geq c_1^2(X,\omega) + 2$ on (X,ω), then if we symplectically rationally blow up a $B_n \hookrightarrow (X,\omega)$, we would obtain a symplectic manifold (X',ω') for which $c_1^2(X',\omega') \leq -1$. As a consequence of a theorem of Taubes [Ta2, Ta4, Ta3], we would then obtain, for a generic ω-compatible almost-complex structure J_ϵ, a J_ϵ-holomorphic embedded sphere Σ_{-1}^ϵ with self-intersection (-1). The consequences of the existence of such a sphere in the symplectic rational blow-up (X',ω') leads to various contradictions of adjunction formulas and results on Seiberg-Witten invariants.

We show that if (X,ω) is such that $n \geq c_1^2(X,\omega) + 2$ (in addition to the two conditions on (X,ω) in Conjecture 1.2), then a symplectic embedding $B_n \hookrightarrow (X,\omega)$ will fall into two types: $\mathcal{A}$ and $\mathcal{E}_k$, $2 \leq k \leq n-1$, (see Definitions 5.3, 5.4, 5.5). The types $\mathcal{A}$ and $\mathcal{E}_k$ are determined by the intersection patterns of a sphere Σ_{-1}, with self-intersection (-1) (obtained as consequence of the sphere Σ_{-1}^ϵ), with the spheres of $C_n \subset (X',\omega')$. We then prove the following theorem:

Theorem D. *(Theorem 5.6) If $B_n \hookrightarrow (X,\omega)$ is a symplectic embedding, where (X,ω) is a symplectic 4-manifold, such that:*

- $b_2^+(X) > 1$,
- $[c_1(X,\omega)] = -[\omega]$ *as cohomology classes,*
- $n \geq c_1^2(X,\omega) + 2$ *and*
- $\mathcal{B}as_X = \{\pm c_1(X,\omega)\}$, *($\mathcal{B}as_X$ denotes the set of Seiberg-Witten basic classes of X,)*

then it cannot be of type $\mathcal{A}$ or of type $\mathcal{E}_k$, $k \geq c_1^2(X,\omega) + 2$.

In Theorem D above, the condition $\mathcal{B}as_X = \{\pm c_1(X,\omega)\}$ on (X,ω) is also true for surfaces of general type.

We also describe a family of symplectic manifolds, $\mathcal{X}$, constructed from the elliptic surfaces $E(m)$, which contain an embedded B_n of type $\mathcal{E}_2$ (not covered by Theorem D), in such a way that

$$n < 3 + \frac{4}{3}c_1^2(X,\omega)\,,$$

for all $(X,\omega) \subset \mathcal{X}$. Thus, also providing evidence for Conjecture 1.2, that every symplectic manifold has a bound on n, above which one can no longer embed a rational homology ball B_n. Both Theorem D and this family of examples suggest that in order for there to exist a symplectic embedding $B_n \hookrightarrow (X,\omega)$ for high n, the manifold (X,ω) needs to at least have a high enough $c_1^2(X,\omega)$.

This paper is organized as follows. In section 2, we give a brief review of Kirby calculus and symplectic/contact structures, which are heavily used in various proofs.

In section 3, we prove Theorem A (Theorem 3.8). First, in section 3.1, we give a detailed description of the rational homology balls B_n. Second, in section 3.2, we give a brief overview of Gompf's Kirby-Stein calculus [Go2]. Third, in section 3.3, using computations of Gompf's Γ invariant (see Theorem 3.5), we show that $(\partial B_n, \xi) \cong (L(n^2, n-1), \xi_{std}) \cong (\partial C_n, \xi')$, where ξ and ξ' are the induced contact structures of

∂B_n and ∂C_n, respectively, and ξ_{std} is the standard contact structure on $L(n^2, n-1)$. Fourth, in section 3.4, we finally give the statement of Theorem A (Theorem 3.8), after stating the definition of the Lagrangian cores $\mathcal{L}_{n,1}$. We show that the neighborhood of the Lagrangian core $\mathcal{L}_{n,1}$ is standard and is symplectomorphic to a symplectic copy of B_n. We conclude the proof, using earlier computations of Gompf's invariant Γ, by showing that after removing the B_ns, we can symplectically glue the C_ns, thus defining the symplectic rational blow-up operation.

In section 4, we prove Theorems B and C (Theorems 4.1 and 4.2), using primarily Kirby calculus. We also draw some additional conclusions from the proofs of these theorems, which leads us to define the notion of "simple" smooth embeddings of B_n into a smooth 4-manifold X.

In section 5, after separating the symplectic embeddings of $B_n \hookrightarrow (X, \omega)$ into types $\mathcal{A}$ and $\mathcal{E}_k$, we prove Theorem D (Theorem 5.6). First, in section 5.2, we provide some additional background material on toric and almost-toric fibrations of symplectic 4-manifolds and Seiberg-Witten invariants. Next, in section 5.3, we prove Theorem D (Theorem 5.6) in four steps, by assuming that there exists a symplectic embedding $B_n \hookrightarrow (X, \omega)$ and obtaining a contradiction. In step 1, section 5.3.1, we show that symplectic embeddings of B_n will indeed be of type $\mathcal{A}$ or $\mathcal{E}_k$. In step 2, section 5.3.2, we construct a cycle γ and compute $c_1(X, \omega) \cdot \gamma$. In step 3, section 5.3.3, we show that if $c_1(X, \omega) \cdot \gamma > 0$ then $\omega \cdot \gamma > 0$, contradicting the $[c_1(X, \omega)] = -[\omega]$ assumption. In step 4, section 5.3.4, we show that if $c_1(X, \omega) \cdot \gamma \leq 0$, then the condition $\mathcal{B}as_X = \{\pm c_1(X, \omega)\}$ or the adjunction formula will be violated. Additionally, in section 5.3.5, we provide explicit examples of symplectic embeddings of $B_n \hookrightarrow (X, \omega)$ of type $\mathcal{E}_2$, which adhere to Conjecture 1.2.

Finally, in section 6, we give some remarks about generalizing the above results to generalized rational homology balls $B_{n,m}$, which are used in the *generalized rational blow-down* operation.

2. Background

2.1. **Review of Kirby calculus.** Kirby calculus is a very useful visual tool for constructing diffeomorphisms between 4- and 3-manifolds. For a full exposition, see [GS] and [OzSt]. We begin by viewing a 4-manifold as a 4-dimensional handlebody. A 4-dimensional k-handle is a copy of $D^k \times D^{4-k}$, attached to the boundary of a 4-manifold along $\partial D^k \times D^{4-k} \cong S^{k-1} \times D^{4-k}$. To build up a 4-manifold as a 4-dimensional handlebody, we start with a 0-handle, a D^4, then attach n 1-handles to it, along $S^0 \times D^3$ in the boundary of the existing 0-handle, as seen in Figure 3. The union of the 0-handle and the n 1-handles will always be diffeomorphic to $\natural S^1 \times D^3$ (here $\natural$ denotes the "boundary sum"). Next, we proceed to attach the 2-handles. Since the 2-handles get attached via $S^1 \times D^3$, it is enough to specify a framed knot in $\natural S^1 \times D^3$, to determine how the 2-handles get attached. If the desired 4-manifold is closed, we do not have to keep track of the 3- and 4-handles, since according to the result of [LP], the gluing of the union of the 3- and 4-handles is unique. Note, the fact that every smooth 4-manifold admits a handle decomposition, comes from Morse theory, with the handles corresponding to the critical points of a Morse function.

Figure 3. Attaching region of a 1-handle

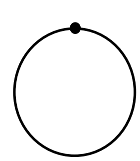


Figure 4. Kirby diagram of a 1-handle

As a result of viewing a 4-manifold as a handlebody, we can represent any 4-manifold by a Kirby diagram. In these diagrams, we depict a 1-handle by a pair of balls $S^0 \times D^3$, as seen in Figure 3 or as a dotted circle, Figure 4, and a 2-handle with a framed knot. The dotted circles are treated as 0-framed unknots. Consequently, in this manner, a 4-manifold can be represented as a link diagram, where some of the components of the link are dotted circles (the 1-handles) and the rest of the components of the link are framed knots (the 2-handles).

If we start out with a Kirby diagram D_1 representing a 4-manifold X_1, and we perform moves (1) – (3) below, and obtain a Kirby diagram D_2 representing a 4-manifold X_2, then X_1 and X_2 are diffeomorphic.

(1) **Isotopies of the link in $\mathbf{S^3}$**. For the Kirby diagram, these are essentially Reidemeister moves for links in S^3 (or $\mathbb{R}^3$).

(2) **Handle slides**. To slide a 2-handle, represented by a knot K_1 over another 2-handle, represented by a knot K_2, one simply replaces K_1 in the Kirby diagram by $K_1 \# K_2'$, the knot connected sum of K_1 and K_2', where K_2' is a push-off of K_2 respecting the framing of K_2. The framing coefficient of the new knot in the diagram, $K_1 \# K_2'$ becomes:

$$fr(K_1 \# K_2') = fr(K_1) + fr(K_2) \pm 2lk(K_1, K_2)$$

where $fr(K_i)$ denotes the framing of K_i. The sign depends on orientations of the knots K_1 and K_2. To perform a 1-handle slide, we simply perform the same action as with a 2-handle slide, treating the dotted circles that represent the 1-handles as 0-framed unknots. This has to be performed with some care, to insure that the resulting 1-handle is still represented by an unknot.

(3) **Adding/deleting a cancelling $1/2$-handle pair (or a cancelling $2/3$-handle pair)**. A cancelling 1/2-handle pair occurs when the dotted circle

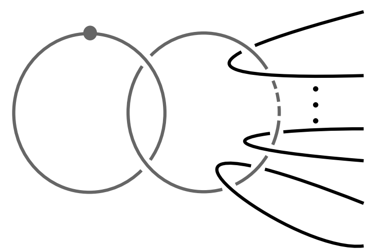


FIGURE 5. A cancelling 1/2-handle pair

representing the 1-handle is linked once (geometrically) to the knot representing the 2-handle, as in Figure 5. A cancelling 2/3-handle pair occurs when the 2-handle is represented by a 0-framed unknot which is disjoint from the rest of the link diagram, and we know that we have a 3-handle in our handlebody description. (Adding a cancelling handle pair amounts to connect summing with a D^4.)

In addition, notice that the above moves $(1)-(3)$ do not change the 3-manifold that is on the boundary of the 4-manifold, ∂X, (also represented by the Kirby diagram). Blowing up a 4-manifold is represented in a Kirby diagram by adding a disjoint (-1)-framed unknot. Clearly, doing this changes the diffeomorphism type of the 4-manifold X, namely to $X\#\overline{\mathbb{C}P^2}$, however, this does not change the 3-manifold on the boundary, ∂X. Similarly, we can construct $X\#\mathbb{C}P^2$ from X by adding a disjoint (1)-framed unknot. As a result, we can construct diffeomorphisms of 3-manifolds, with Kirby diagrams, using moves $(1)-(3)$ above and adding/deleting (± 1)-framed unknots.

2.2. **Review of symplectic and contact structures.** Here, we will briefly review symplectic, contact, and almost-complex structures. For a full exposition of symplectic geometry and topology, we refer the reader to [MS2, Ca]. For a full exposition of contact geometry, we refer the reader to [Ge] and for an additional reference for Legendrian knots, see [Et]. In addition, [OzSt] is an excellent source for the interaction of symplectic and contact topology, as well as Stein surfaces.

2.2.1. *Symplectic structures.* We will discuss symplectic structures on 4-manifolds, although in general symplectic structures can be defined on $2n$-dimensional manifolds.

Definition 2.1. A *symplectic 4-manifold* (X, ω) is a 4-dimensional smooth manifold X equipped with a *nondegenerate* and *closed* 2-form ω.

A 2-form $\omega \in \Omega^2(X)$ on a smooth 4-manifold X is *nondegenerate* if it is nondegenerate on every tangent space T_pX, $\forall p \in X$, meaning that for every nonzero vector $u \in T_pX$, $\exists v \in T_pX$ such that $\omega(u, v) \neq 0$. This property of the 2-form implies that $\omega^2 = \omega \wedge \omega$ is nowhere 0.

Definition 2.2. A *symplectomorphism* between two symplectic 4-manifolds (X_1, ω_1) and (X_2, ω_2) is a diffeomorphism $\phi : X_1 \to X_2$ preserving the symplectic form ω, i.e. $\omega_1 = \phi^*(\omega_2)$.

Similarly to a Riemannian metric, the nondegeneracy of a symplectic form gives an isomorphism between TX and T^*X. However, very much unlike a Riemannian metric, locally, every 4-dimensional symplectic manifold is symplectomorphic to $(\mathbb{R}^4, \omega_0)$, where

$$\omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$$

is the standard symplectic structure on $\mathbb{R}^4$, with respect to the standard coordinates (x_1, y_1, x_2, y_2).

Symplectic structures on a 4-manifold are very closely related to almost-complex structures.

Definition 2.3. An *almost-complex* structure of a smooth 4-manifold X, is an endomorphism $J : TX \to TX$, such that $J^2 = -\mathrm{Id}_{TX}$.

We say that a 2-form ω *tames* an almost-complex structure J if $\omega(v, Jv) > 0$ for all $v \neq 0$ in TX. We say that a 2-form ω is *compatible* with an almost-complex structure

J if in addition to taming J, we have $\omega(Jv_1, Jv_2) = \omega(v_1, v_2)$ for all $v_1, v_2 \in TX$. We have the following important result (See [GS] for example):

Proposition 2.4. *Any symplectic 4-manifold (X, ω) admits a compatible almost-complex structure J. Moreover, the space of compatible almost-complex structures is contractible.*

A 2-dimensional *symplectic submanifold* $\Sigma \subset X$, is a 2-dimensional submanifold of X, such that $\omega_{|\Sigma}$ is a nondegenerate, closed 2-form on Σ. A 2-dimensional *Lagrangian submanifold* $\Sigma \subset X$, is a 2-dimensional submanifold of X, such that $\omega_{|\Sigma} = 0$, if in addition Σ is closed and oriented, then we always have $-\chi(\Sigma) = [\Sigma]^2$. Consequently, the neighborhood of a Lagrangian submanifold, in a symplectic 4-manifold, is always "standard". If J is an almost-complex structure on M, then a *pseudo-holomorphic submanifold* (real 2-dimensional) $\Sigma \subset X$ is a 2-dimensional submanifold of X such that J maps $T\Sigma \subset TX$ into itself. For closed pseudo-holomorphic submanifolds, we always have:

$$-\chi(\Sigma) = [\Sigma]^2 - c_1(X, \omega)[\Sigma]. \tag{2.1}$$

Note, $[x]$ will always denote the (co)homology class of x.

2.2.2. *Contact structures.* Closely related to symplectic 4-dimensional manifolds are 3-dimensional contact manifolds. Note, although contact structures can be defined for all odd-dimensional manifolds, we will focus here only on 3-dimensional manifolds.

Definition 2.5. A *contact structure* on a 3-manifold M is a plane field ξ (i.e. a subbundle of the tangent bundle TM such that $\forall p \in M, T_pM \subset \xi$ is a 2-dimensional subspace of T_pM), such that if α is any 1-form for which $\xi = \ker \alpha$, then $\alpha \wedge d\alpha \neq 0$.

Definition 2.6. There is a *contactomorphism* between two contact structures ξ_1 and ξ_2 on M, if there is a diffeomorphism $\phi : M \to M$ such that $\phi_*(\xi_1) = \xi_2$.

Contact structures on 3-dimensional manifolds are similar to symplectic structures on 4-dimensional manifolds, since just like symplectic structures, contact structures are all locally contactomorphic. The "standard"' contact structure on $\mathbb{R}^3$, with Cartesian coordinates (x, y, z) is given by the 1-form:

$$\alpha = dz + xdy$$

where $\xi = \left\{ \frac{\partial}{\partial x}, x\frac{\partial}{\partial z} - \frac{\partial}{\partial y} \right\}$ are the contact planes. The "standard"' contact structure on $S^1 \times S^2 \subset S^1 \times \mathbb{R}^3$, with coordinates (θ, x, y, z) is given by the contact 1-form:

$$\alpha = zd\theta + xdy - ydx.$$

Given a contact 3-manifold (M, ξ), one can define knots in M that respect the contact structure in a certain way.

Definition 2.7. A knot K in a contact manifold (M, ξ), is called *Legendrian* if it is always tangent to the contact planes, i.e. $T_pK \subset \xi_p$, $\forall p \in K$.

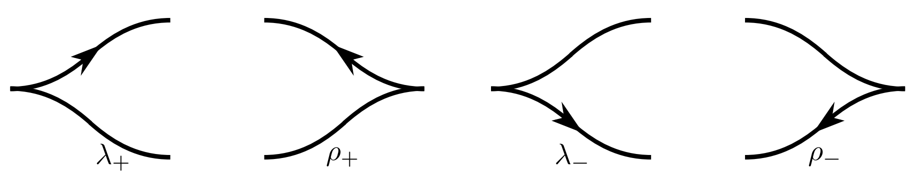


FIGURE 6. Cusps in a front projection of a Legendrian knot

One can compute the so called "classical" invariants of a Legendrian knot K: *Thurston-Bennequin invariant*, $tb(K)$ and the *rotation number*, $rot(K)$. If K is a null homologous knot, then $tb(K)$ is the linking number $\ell k(K, K')$, where K' is a push-off of K by a vector field along K which is transverse to the contact planes ξ. The rotation number of a null-homologous knot K, is the winding number of TK after a certain trivialization of ξ along K. If $K \in \mathbb{R}^3$ (or S^3), then we can compute $tb(K)$

and $rot(K)$ easily from the *front projection* (projection to the $yz-$plane) of a knot. In this projection, we don't have any vertical tangencies but we do have cusps. If we give a Legendrian knot an orientation, then from the number of upward left cusps (λ_+), downward left cusps (λ_-), upward right cusps (ρ_+) and downward right cusps (ρ_-) (see Figure 6) we can compute $tb(K)$ and $rot(K)$ as follows:

$$tb(K) = w(K) - \frac{1}{2}(\lambda(K) + \rho(K)) = w(K) - \lambda(K) \tag{2.2}$$

$$rot(K) = \lambda_- - \rho_+ = \rho_- - \lambda_+ \tag{2.3}$$

where $w(K)$ is the writhe of K.

Contact 3-manifolds can be divided into two categories: *overtwisted* and *tight* (see [El1]).

Definition 2.8. A contact 3-manifold (M, ξ) is *overtwisted* if it contains an *overtwisted* disk (an embedded disk $D \subset (M, \xi)$ with $\partial D = K$ a Legendrian knot with $tb(K) = 0$). A contact 3-manifold (M, ξ) is *tight* if it does not contain an overtwisted disk.

A lot of literature has been concerned with classifying tight contact structures for a given contact 3-manifold. A result of Honda [Ho] that will be relevant to us later, is that there are $|(r_0 + 1)(r_1 + 1) \cdots (r_k + 1)|$ tight contact structures on the lens space $L(p, q)$, where $[r_0, r_1, \ldots, r_k]$ is the continued fraction expansion of $-\frac{p}{q}$.

2.2.3. *Relationship between contact and symplectic structures.* As mentioned before, symplectic 4-manifolds and contact 3-manifolds are closely related. For example, there are several ways that a symplectic 4-manifold can have a contact 3-manifold as its boundary. A contact 3-manifold (M, ξ) is *weakly symplectically fillable* (or *fillable*) if there exists a compact symplectic 4-manifold (X, ω) such that $\partial X = M$ and $\omega_{|_\xi} \neq 0$. A contact 3-manifold (M, ξ) is *strongly symplectically fillable* if in addition to *fillable*, ω is exact near the boundary, $\omega = d\alpha$, such that $\ker(\alpha_{|\partial W}) = \xi$ (in other words, (M,ξ) is an ω-convex boundary of (X,ω)). A contact 3-manifold (M,ξ) is *holomorphically fillable* if there exists a compact complex surface (X,J) such that (M,ξ) is contactomorphic to the contact structure on ∂X given by complex tangencies. A contact 3-manifold (M,ξ) is *Stein fillable* if it is the J-convex boundary of a Stein surface [OzSt]. For discussion on Stein surfaces, see section 3.2.

From a contact 3-manifold (M,ξ) we can construct the symplectic 4-manifold $Symp(M,\xi)$, called the *symplectization* of (M,ξ), which is defined as follows:

$$Symp(M,\xi) = \left\{ v \in T_p^*M | v = t\alpha_p \, \text{for some} \, t > 0 \right\} , \tag{2.4}$$

where α is a chosen contact 1-form for ξ. The 4-manifold $Symp(M,\xi)$ is diffeomorphic to $M \times (0,\infty)$ equipped with a symplectic 2-form

$$\omega = t d\alpha + dt \wedge \alpha$$

(note, this construction is independent of the choice of α). Additionally, given a symplectic 4-manifold with a contact 3-manifold on its boundary $\partial(X,\omega) = (M,\xi)$ (symplectic filling), we can construct the *symplectic completion* $(X,\omega)^+$ of (X,ω) by

$$(X,\omega)^+ = (X,\omega) \cup Symp(M,\xi). \tag{2.5}$$

3. Symplectic rational blow-up

In the regular symplectic blow-up operation, we take a standard symplectic ball D^4, of a certain chosen radius, (a Darboux neighborhood of a point), and remove it while collapsing its boundary S^3, by the Hopf map $h : S^3 \to S^2$, onto S^2. The sphere S^2 will have self-intersection (-1), i.e. $[S^2]^2 = -1$. In order to define a symplectic rational blow-up, our first step is to endow the rational homology balls B_n with a symplectic structure such that B_n and ∂B_n are totally standard, to guarantee that it "matches up" with ∂C_n. Consequently, we want to find a symplectic structure ω_n on B_n such that $\partial(B_n, \omega_n) = (L(n^2, n-1), \xi_{std})$.

3.1. **Description of the rational homology balls B_n.** There are several ways to give a description of the rational homology balls B_n. One of them is a Kirby calculus diagram seen in Figure 2. This represents the following handle decomposition: Start with a 0-handle, a standard 4-disk D^4, attach to it a 1-handle $D^1 \times D^3$. Call the resultant space X_1, it is diffeomorphic to $S^1 \times D^3$ and has boundary $\partial X_1 = S^1 \times S^2$. Finally, we attach a 2-handle $D^2 \times D^2$. The boundary of the core disk of the 2-handle gets attached to the closed curve, K, in ∂X_1 which wraps n times around the $S^1 \times *$ in $S^1 \times S^2$. We can also represent B_n by a slightly different Kirby diagram, which is more cumbersome to manipulate but is more visually informative, as seen in Figure 7, where the 1-handle is represented by a pair of balls.

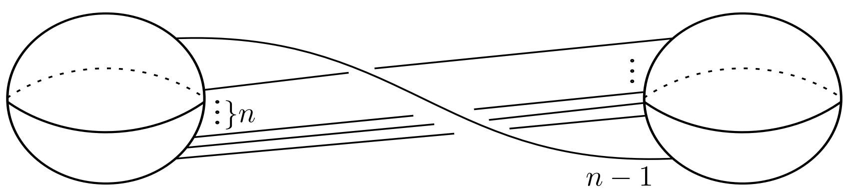


FIGURE 7. **Another Kirby diagram of B_n**

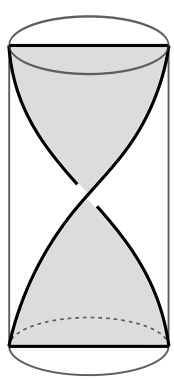


FIGURE 8. L_2'

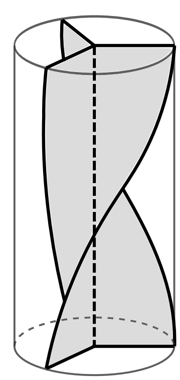


FIGURE 9. L_3'

The rational homology ball B_2 can also be described as an unoriented disk bundle over $\mathbb{R}P^2$. Since $\mathbb{R}P^2$ is the union of a Mobius band M and a disk D, we can visualize $\mathbb{R}P^2$ sitting inside B_2, with the Mobius band and its boundary $(M, \partial M)$ embedded in $(X_1 \cong S^1 \times D^3, \partial X_1 \cong S^1 \times S^2)$ (Figure 8, with the ends of the cylinder identified), and the disk D as the core disk of the attaching 2-handle. We will construct something similar for $n \geq 3$. Instead of the Mobius band sitting inside X_1, as for $n = 2$, we have a "n-Mobius band", L'_n, sitting inside X_1. The case of $n = 3$ is illustrated in Figure 9, again with the ends of the cylinder identified. In other words, L'_n is a singular surface, homotopic to a circle, in $X_1 \cong S^1 \times D^3$, whose boundary is the closed curve K in $\partial X_1 \cong S^1 \times S^2$, and it includes the circle, $S = S^1 \times 0$ in $S^1 \times D^3$. Let $L_n = L'_n \cup_K D$, where D is the core disk of the attached 2-handle (along K). We will call L_n the core of the rational homology ball B_n; observe, that $L_2 \cong \mathbb{R}P^2$.

The cores L_n will be used as geometrical motivation in the construction of a symplectic structure on the rational homology balls B_n. For $n = 2$, if we have an embedded $\mathbb{R}P^2$ in (X, ω), such that $\omega|_{\mathbb{R}P^2} = 0$, (i.e. a Lagrangian $\mathbb{R}P^2$) then the $\mathbb{R}P^2$ will have a totally standard neighborhood, which will be symplectomorphic to the rational homology ball B_2. The symplectic structures which we will endow on the rational homology balls B_n will have the cores $L_n \hookrightarrow B_n$ be Lagrangian, which we will refer to later as $\mathcal{L}_{n,1}$ in section 3.4.

3.2. **Kirby-Stein calculus.** We will use Eliashberg's Legendrian surgery construction [El2] along with Gompf's handlebody constructions of Stein surfaces [Go2] to put symplectic structures on the B_ns, which will be induced from Stein structures. We will give a brief overview of the aforementioned constructions, beginning with a theorem of Eliashberg's on a 4-manifold admitting a Stein structure [El2] [Go2]:

Theorem 3.1. *A smooth, oriented, open 4-manifold X admits a Stein structure if and only if it is the interior of a (possibly infinite) handlebody such that the following hold:*

(1) *Each handle has index* ≤ 2*,*

(2) *Each 2-handle* h_i *is attached along a Legendrian curve* K_i *in the contact structure induced on the boundary of the underlying 0- and 1-handles, and*

(3) *The framing for attaching each* h_i *is obtained from the canonical framing on* K_i *by adding a single left (negative) twist.*

A smooth, oriented, compact 4-manifold X *admits a Stein structure if and only if it has a handle decomposition satisfying (1), (2), and (3). In either case, any such handle decomposition comes from a strictly plurisubharmonic function (with* ∂X *a level set).*

From Theorem 3.1, it follows that if we wanted to construct a Stein surface S, such that its strictly plurisubharmonic Morse function did not have any index 1 critical points, then all we have to do to give a handlebody description of S is to specify a Legendrian link L in $S^3 = \partial B^4 = \partial$(0-handle), and attach 2-handles with framing $tb(K_i)-1$, where K_i are the components the framed link L. If we allow index 1 critical points, then we must include 1-handles in the handlebody decomposition of S. If a handle decomposition of a compact, oriented 4-manifold has only handles with index 0, 1, or 2, then all that one needs to specify it is a framed link in $\#mS^1 \times S^2 = \partial$ (0-handle $\cup$ 1-handles). Consequently, in order to deal with arbitrary Stein surfaces, Gompf [Go2] established a standard form for Legendrian links in $\#mS^1 \times S^2$:

Definition 3.2. ([Go2], definition 2.1) A *Legendrian link diagram* in *standard form*, with $m \geq 0$ 1-handles, is given by the following data (see Figure 10):

(1) A rectangular box parallel to the axes in $\mathbb{R}^2$,

(2) A collection of m distinguished segments of each vertical side of the box, aligned horizontally in pairs and denoted by balls, and

(3) A front projection of a generic Legendrian tangle (i.e. disjoint union of Legendrian knots and arcs) contained in the box, with endpoints lying in the distinguished segments and aligned horizontally in pairs.

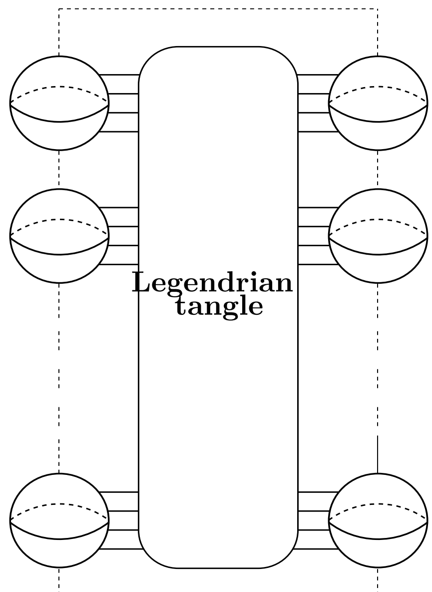


FIGURE 10. **Legendrian link diagram in standard form**

All one needs to do is attach 1-handles to each pair of balls and one gets a link in $\#mS^1 \times S^2$. Using this definition, Gompf [Go2] establishes a full list of Kirby-Legendrian calculus type moves that will relate any two such diagrams:

Theorem 3.3. *(*[Go2]*, theorem 2.2) Let H denote a handlebody consisting of a 0-handle and m 1-handles (ordered), with the canonical Stein structure determined by Theorem 3.1 and let ξ be a contact structure on ∂H. The boundary of H can be identified with a contact manifold obtained from the standard contact structure on S^3 by removing smooth balls and gluing the resulting boundaries as in Figure 10. The identification exhibits H in the usual way as a smooth handlebody, and it can be assumed to match the above data for H with corresponding pre-assigned data for the diagram. Any Legendrian link in $\partial H = \#mS^1 \times S^2$ is contact isotopic to one in* *standard form. Two Legendrian links in standard form are contact isotopic in ∂H if and only if they are related by a sequence of the six moves shown in (Figures 3 and 9 in* [Go2]*), together with isotopies of the box that fix the boundary outside of the balls and introduce no vertical tangencies.*

The classical invariants of Legendrian knots, such as the Thurston Bennequin number $tb(K)$ and the rotation number $rot(K)$ still make sense for the Legendrian link diagrams in standard form, although with a few caveats. Both $tb(K)$ and $rot(K)$ can be computed for a knot K that's part of a Legendrian link diagram as in Figure 10 from the same formulas as in a standard front projection of Legendrian knots in $\mathbb{R}^3$ (also see Figure 6):

$$tb(K) = w(K) - \frac{1}{2}(\lambda(K) + \rho(K)) = w(K) - \lambda(K) \tag{3.1}$$

$$rot(K) = \lambda_- - \rho_+ = \rho_- - \lambda_+ \, . \tag{3.2}$$

The invariant $rot(K)$ doesn't change under the moves in Theorem 3.3. However, one of the moves changes $tb(K)$ by twice the number of times (with sign) that K runs over the 1-handle (involved in the move). The change is due to how it is obtained from the diagram and not the canonical framing. Moreover, it is shown that in these Legendrian diagrams (as in Figure 10) the number $tb(K) + rot(K) + 1$ is always congruent modulo 2 to the number of times that K crosses the 1-handles.

Putting together the Legendrian link diagrams in standard form, the classical Legendrian knot invariants that can be read from them, the complete list of their Kirby calculus moves in Theorem 3.3, and Eliashberg's Theorem 3.1, the following characterization of compact Stein surfaces with boundary can be made:

Proposition 3.4. [Go2] *A smooth, oriented, compact, connected 4-manifold X admits the structure of a Stein surface (with boundary) if and only if it is given by a* *handlebody on a Legendrian link in standard form (Definition 3.2) with the i'th 2-handle* h_i*, attached to the i'th link component* K_i*, with framing* $tb(K_i) - 1$ *(as given by Formula 3.1). Any such handle decomposition is induced by a strictly plurisubharmonic function. The Chern class* $c_1(J) \in H^2(X;\mathbb{Z})$ *of such a Stein structure* J *is represented by a cocycle whose value on each* h_i*, oriented as in Theorem 3.1, is* $rot(K_i)$ *(as given by Formula 3.2).*

The benefits of these Legendrian link diagrams, is that one can compute several useful invariants of the Stein surface and its boundary straight from them. In particular, Gompf ([Go2], section 4) gave a complete set of invariants of 2-plane fields on 3-manifolds, up to their homotopy classes, which, in particular, could be used to distinguish contact structures of the boundaries of Stein surfaces. We will describe one such invariant, Γ, which we will later use in section 3.3. In general, the classification of 2-plane fields on an oriented 3-manifold M is equivalent to fixing a trivialization of the tangent bundle TM and classifying maps $\varphi : M \to S^2$ up to homotopy, which was done in [Po]. Γ is an invariant of 2-plane fields on closed, oriented 3-manifolds, that is a 2-dimensional obstruction, thus it measures the associated *spin*^{c} structure. The advantage of Γ is that it can be specified without keeping explicit track of the choice of trivialization of TM, and instead can be measured in terms of spin structures of the 3-manifold M.

In order to define Γ we need to establish some notation and terminology. Let (X, J) be a Stein surface with a Stein structure J. There is a natural way to obtain a contact structure ξ on its boundary $\partial X = M$, by letting ξ be the field of complex lines in $TM \subset TX|_M$, in other words

$$\xi = T\partial X \cap JT\partial X\,.$$

Assume X can be presented in standard form, as in Figure 10. We can construct a manifold X^*, which is obtained from X by surgering out all of the 1-handles of

X (this can be done canonically). As a result, we have $\partial X = \partial X^* = M$, and X^* can be described by attaching 2-handles along a framed link L in $\partial B^4 = S^3$, which can be obtained by gluing the lateral edges of the box in Figure 10. The 1-handles of X become 2-handles of X^* that are attached along unknots with framing 0, call this subset of links $L_0 \subset L$. The 2-handles of X remain 2-handles of X^*, with the same framing. Since Γ will be defined in terms of the spin structures of M, it is useful to express the spin structures of M as characteristic sublinks of L; thus, for each $\mathfrak{s} \in Spin(M)$, we will associate a characteristic sublink $L(\mathfrak{s}) \subset L$. Recall, that L' is a *characteristic sublink* of L if for each component K of L, the framing of K is congruent modulo 2 to $\ell k(K, L')$ [Go2] [GS]. (Note, here $\ell k(A, B)$ is the usual linking number if $A \neq B$, and the framing of A if $A = B$, and is extended bilinearly if A or B have more than one component.) Finally, we can define Γ for a boundary of a compact Stein surface, by a formula obtained from a diagram of the Stein surface in standard form:

Theorem 3.5. *(Gompf* [Go2]*, Theorem 4.12) Let X be a compact Stein surface in standard form, with $\partial X = (M, \xi)$, and X^*, $L = K_1 \cup \ldots \cup K_m$ and L_0 as defined above. Let $\{\alpha_1, \ldots, \alpha_m\} \subset H_2(X^*; \mathbb{Z})$ be the basis determined by $\{K_1, \ldots, K_m\}$. Let $\mathfrak{s}$ be a spin structure on M, represented by a characteristic sublink $L(\mathfrak{s}) \subset L$. Then $PD\Gamma(\xi, \mathfrak{s})$ is the restriction to M of the class $\rho \in H^2(X^*; \mathbb{Z})$ whose value on each α_i is the integer*

$$\langle \rho, \alpha_i \rangle = \frac{1}{2}(rot(K_i) + \ell k(K_i, L_0 + L(\mathfrak{s}))) \,, \tag{3.3}$$

(note: $rot(K_i)$ is defined to be 0 if $K_i \subset L_0$.)

3.3. **Computations of Gompf's invariant Γ.** In this section, we compute Gompf's Γ invariant for $(L(n^2, n-1), \xi_{std})$, (section 3.3.1), $\partial(B_n, J_n)$, (section 3.3.2), and for $\partial(C_n, J_n')$, (section 3.3.3). (The almost-complex structures J_n and J_n' will be defined later.) Note, by the standard contact structure ξ_{std} on $(L(n^2, n-1)$, we mean the contact structure that descends to $L(n^2, n-1)$ from the standard contact structure on S^3, via the identification $L(n^2, n-1) = S^3/G_{n^2,n-1}$, where $G_{n^2,n-1}$ is the subgroup

$$G_{n^2,n-1} = \left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{n-1} \end{pmatrix} \,|\, \zeta^{n^2} = 1 \right\} \subset U(2)\,.$$

3.3.1. *Computations of* Γ *for* $(L(n^2, n-1), \xi_{std})$. In 2006, Lisca [Lis] classified all the symplectic fillings of $(L(p,q), \xi_{std})$ up to diffeomorphisms and blow-ups. In order to show that the boundaries of the symplectic 4-manifolds he constructed are the lens spaces with the standard contact structure $(L(p,q), \xi_{std})$, he computed the Gompf invariant Γ of $(L(p,q), \xi_{std})$ by expressing the contact manifold as the link of a cyclic quotient singularity. We will use his calculations, in the case of $p = n^2$ and $q = n-1$, to match up to our own calculations of Γ for $\partial(B_n, \omega_n)$ and $\partial(C_n, \omega'_n)$.

As mentioned above, $(L(n^2, n-1), \xi_{std})$ can be expressed as a link of a cyclic quotient singularity. There is a canonical resolution of this singularity with an exceptional divisor, with a neighborhood $R_{n^2,n-1}$. Let $l_1 \cup l_2$ be the union of two distinct complex lines in $\mathbb{C}P^2$. After successive blow-ups, we can obtain a string C of rational curves in $\mathbb{C}P^2 \# N\overline{\mathbb{C}P^2}$ of type $(1, -1, -2, \ldots, -2, -n)$ (with $(n-1)$ of -2's), with $\nu(C)$ a regular neighborhood of C. It is shown in ([Lis], section 6) that there is a natural orientation preserving diffeomorphism from the complement of $\nu(C)$ to $R_{n^2,n-1}$. The boundary of $\nu(C)$ is an oriented 3-manifold which can be given by a surgery presentation of unknots U_0, ..., U_{n+1} (Figure 11), where $\nu_0, \ldots, \nu_{n+1}$ are the generators of $H_1(\partial\nu(C); \mathbb{Z})$. If the unknot U_0 is blown down, we have a natural identification

$$\nu(C) = -L(n^2, n-1) = L(n^2, n^2 - (n-1)) \tag{3.4}$$

since $[-2,-2,\dots,-2,-n]$, with n amount of (-2)s, is the continued fraction expansion of $\frac{n^2}{n^2-(n-1)}$.

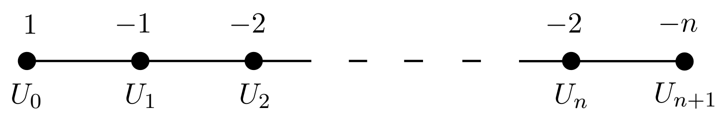


FIGURE 11. **Surgery diagram of $\partial\nu(C)$**

The relations of $\nu_0,\dots,\nu_{n+1}$ in $H_1(\partial\nu(C);\mathbb{Z})$ are as follows:

$$\left.\begin{aligned}
&\nu_0+\nu_1=0\\
&\nu_0-\nu_1+\nu_2=0\\
&\nu_1-2\nu_2+\nu_3=0\\
&\nu_2-2\nu_3+\nu_4=0\\
&\vdots\\
&\nu_{n-1}-2\nu_n+\nu_{n+1}=0\\
&\nu_n-n\nu_{n+1}=0
\end{aligned}\right\}\Longrightarrow
\begin{aligned}
&\nu_0=-\nu_1\\
&\nu_2=2\nu_1\\
&\nu_3=3\nu_1\\
&\nu_4=4\nu_1\\
&\vdots\\
&\nu_{n+1}=(n+1)\nu_1\\
&(n^2)\nu_1=0\,.
\end{aligned}$$

Lisca applied a slight generalization of Theorem 3.5 ([Lis], theorem 6.2), and computed the value of Gompf's Γ invariant of $\partial\nu(C)=-L(p,q)$. For our purposes we restate it with $p=n^2$ and $q=n-1$, and we will handle the even and odd values of n seperately.

For n odd, the lens spaces $L(n^2,n^2-(n-1))$ each have one spin structure $\mathfrak{t}$, which can be specified by the characteristic sublink $L(\mathfrak{t})=U_1\cup U_3\cup U_5\cup\ldots\cup U_n$. Also,

$L_0 = \emptyset$, (see (3.3)). Consequently, we have:

$$\begin{aligned}
PD\Gamma_{L(n^2,n-1)}(\xi_{std}, \mathfrak{t}) &= -PD\Gamma_{L(n^2,n^2-(n-1))}(\xi_{std}, \mathfrak{t}) \\
&= -\nu_0 - \nu_1 + \nu_2 - \nu_3 + \ldots - \nu_n + \frac{n-1}{2}\nu_{n+1} \\
&= \nu_1 - \nu_1 + 2\nu_1 - 3\nu_1 + \cdots - n\nu_1 + \frac{n^2-1}{2}\nu_1 \\
&\equiv \frac{n^2-n}{2}\nu_1 \mod n^2 .
\end{aligned}$$

For n even, the lens spaces $L(n^2, n^2-(n-1))$ each have two spin structures $\mathfrak{t}_1$ and $\mathfrak{t}_2$, corresponding to the characteristic sublinks $L(\mathfrak{t}_1) = U_0$ and $L(\mathfrak{t}_2) = U_1 \cup U_3 \cup \ldots \cup U_{n+1}$ respectively. As before, $L_0 = \emptyset$. Consequently we have:

$$\begin{aligned}
PD\Gamma_{L(n^2,n-1)}(\xi_{std}, \mathfrak{t}_1) &= -PD\Gamma_{L(n^2,n^2-(n-1))}(\xi_{std}, \mathfrak{t}_1) \\
&= -\nu_0 + \nu_{n+1} \\
&= \nu_1 + \frac{n^2-n-2}{2}\nu_1 \\
&\equiv \frac{n^2-n}{2}\nu_1 \mod n^2 \\
PD\Gamma_{L(n^2,n-1)}(\xi_{std}, \mathfrak{t}_2) &= -PD\Gamma_{L(n^2,n^2-(n-1))}(\xi_{std}, \mathfrak{t}_2) \\
&= -\nu_0 - \nu_1 + \nu_2 - \nu_3 + \ldots + \nu_{n+1} \\
&= \nu_1 - \nu_1 + 2\nu_1 - 3\nu_1 + 4\nu_1 - \ldots - (n+1)\nu_1 \\
&\equiv \frac{n}{2}\nu_1 \mod n^2 .
\end{aligned}$$

3.3.2. *Computations of* Γ *for* $\partial(B_n, J_n)$. Next, we will put Stein structures J_n on the rational homology balls B_n, and compute the Γ invariant on their boundaries. We will present (B_n, J_n) as a Legendrian diagram in standard form, as in Definition 3.2. However, before that can be done we must first express the B_ns with a slightly different Kirby diagram, one that has appropriate framings with which its 2-handles are attached, thus enabling us to put it in Legendrian standard form. Figure 12 shows another Kirby diagram of B_n, that is equivalent to the one in Figure 2 and Figure 7, by a series of Kirby moves seen in Appendix A.

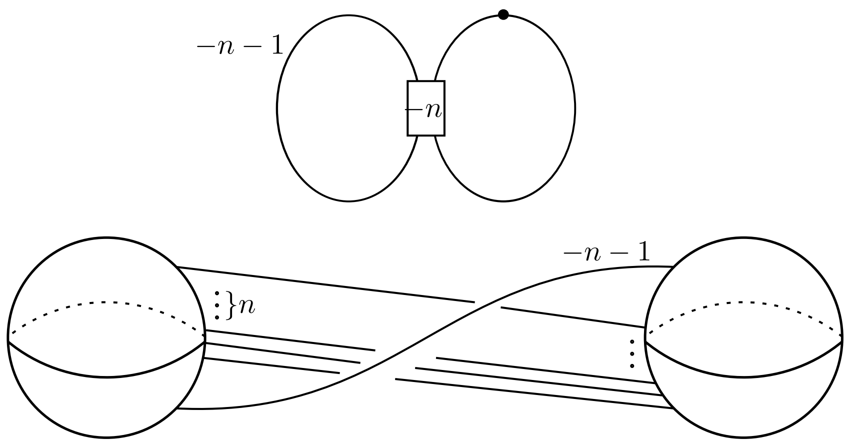


Figure 12. **Another Kirby diagram of B_n**

Having this Kirby diagram for B_n, we are now ready to put it in Legendrian standard form, as seen in Figure 13. The orientation was chosen arbitrarily, but will remain fixed throughout. Observe, that the Legendrian knot K_2^n in the diagram has the following classical invariants:

$$tb(K_2^n) = w(K_2^n) - \lambda(K_2^n) = -(n-1) - 1 = -n$$

$$rot(K_2^n) = \lambda_- - \rho_+ = 1\,.$$

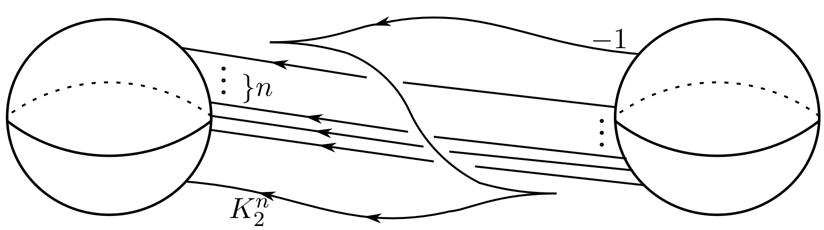


Figure 13. **Kirby diagram of B_n with Stein structure J_n**

Therefore, the framing with which the 2-handle is attached is precisely as dictated by Theorem 3.4, namely $tb(K_2^n) - 1 = -n - 1$.

Now that we described the Stein structure J_n on B_n, we are ready to compute the Γ invariant of $\partial(B_n, J_n) = (\partial B_n, \xi)$ where ξ is the induced contact structure, $\xi = T\partial B_n \cap JT\partial B_n$. Recall, that since the set of Stein structures of a 4-manifold is a subset of the set of almost-complex structures of a 4-manifold, then from the Stein surface (B_n, J_n) we naturally get a symplectic 4-manifold (B_n, ω_n), where the symplectic form ω_n is induced by the almost-complex structure J_n. As a result, (B_n, ω_n) is a symplectic filling of $(\partial B_n, \xi) = (L(n^2, n-1), \xi)$. Our goal is to show that $\xi \cong \xi_{std}$.

As described in Theorem 3.5, we construct the manifold B_n^* from B_n, where we replace the 1-handle in B_n with a 2-handle attached to an unknot with framing 0. A diagram for B_n^* is seen in Figure 14.

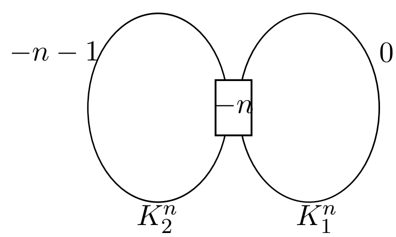


FIGURE 14. **Kirby diagram of B_n^***

Let μ_1 and μ_2 be the meridians of the knots K_1^n and K_2^n, as depicted in Figure 14. Let α_1, α_2 be the basis of $H_2(B_n^*; \mathbb{Z})$ determined by K_1^n and K_2^n. By definition, we have $rot(K_1^n) = 0$, and according to the Stein structure J_n, we have $rot(K_1^n) = 1$. The relations of μ_1 and μ_2 in $H_1(\partial B_n; \mathbb{Z})$ are:

$$\left.\begin{aligned} &-n\mu_2 = 0 \\ &-n\mu_1 - (n+1)\mu_2 = 0 \end{aligned}\right\} \Longrightarrow \quad \begin{aligned} &\mu_2 = -n\mu_1 \\ &(n^2)\mu_1 = 0\,. \end{aligned}$$

For n odd, as before, $\partial B_n = L(n^2, n-1)$ has only one spin structure, $\mathfrak{s}$, whose characteristic sublink is $L(\mathfrak{s}) = \emptyset$. Additionally, we have $L_0 = K_1^n$. Letting ρ be as in Theorem 3.5, we have:

$$\begin{aligned}\langle \rho, \alpha_1 \rangle &= \frac{1}{2}(rot(K_1^n) + \ell k(K_1^n, K_1^n)) = \frac{1}{2}(0+0) = 0 \\ \langle \rho, \alpha_2 \rangle &= \frac{1}{2}(rot(K_2^n) + \ell k(K_2^n, K_1^n)) = \frac{1}{2}(1-n)\,.\end{aligned}$$

Using the above, we compute $PD\Gamma_{\partial B_n}(\xi, \mathfrak{s})$:

$$\begin{aligned}PD\Gamma_{\partial B_n}(\xi, \mathfrak{s}) &= \langle \rho, \alpha_1 \rangle \mu_1 + \langle \rho, \alpha_2 \rangle \mu_2 \\ &= 0\mu_1 + \frac{1-n}{2}\mu_2 \\ &\equiv \frac{n^2-n}{2}\mu_1 \mod n^2\,.\end{aligned}$$

For n even, $\partial B_n = L(n^2, n-1)$ has two spin structures $\mathfrak{s}_1$ and $\mathfrak{s}_2$, corresponding to the characteristic sublinks $L(\mathfrak{s}_1) = K_2^n$ and $L(\mathfrak{s}_2) = K_1^n + K_2^n$ respectively, (and $L_0 = K_1^n$ as before). We have for the spin structure $\mathfrak{s}_1$:

$$\begin{aligned}\langle \rho, \alpha_1 \rangle &= \frac{1}{2}(rot(K_1^n) + \ell k(K_1^n, K_1^n + K_2^n)) = \frac{1}{2}(0-n) = \frac{-n}{2} \\ \langle \rho, \alpha_2 \rangle &= \frac{1}{2}(rot(K_2^n) + \ell k(K_2^n, K_1^n + K_2^n)) = \frac{1}{2}(1-(2n+1)) = -n\,.\end{aligned}$$

Therefore,

$$\begin{aligned}PD\Gamma_{\partial B_n}(\xi, \mathfrak{s}_1) &= \langle \rho, \alpha_1 \rangle \mu_1 + \langle \rho, \alpha_2 \rangle \mu_2 \\ &= \frac{-n}{2}\mu_1 - n\mu_2 \\ &\equiv \frac{2n^2-n}{2}\mu_1 \mod n^2\,.\end{aligned}$$

For the spin structure $\mathfrak{s}_2$ we get:

$$\begin{aligned}\langle \rho, \alpha_1 \rangle &= \frac{1}{2}(rot(K_1^n) + \ell k(K_1^n, 2K_1^n + K_2^n)) = \frac{1}{2}(0-n) = \frac{-n}{2} \\ \langle \rho, \alpha_2 \rangle &= \frac{1}{2}(rot(K_2^n) + \ell k(K_2^n, 2K_1^n + K_2^n)) = \frac{1}{2}(1-(3n+1)) = \frac{-3n}{2}\,.\end{aligned}$$

Therefore,

$$
\begin{aligned}
PD\Gamma_{\partial B_n}(\xi, \mathfrak{s}_2) &= \langle \rho, \alpha_1 \rangle \, \mu_1 + \langle \rho, \alpha_2 \rangle \, \mu_2 \\
&= \frac{-n}{2}\mu_1 - \frac{3n}{2}\mu_2 \\
&\equiv \frac{n^2 - n}{2}\mu_1 \mod n^2 \, .
\end{aligned}
$$

3.3.3. *Computations of* Γ *for* $\partial(C_n, J_n')$. Next, we will compute the Γ invariant for $\partial(C_n, J_n')$, where J_n' is a Stein structure on C_n. We will exhibit the explicit Stein structure on C_n with a Legendrian link diagram (with no 1-handles) which induces $\xi' = T\partial C_n \cap JT\partial C_n$ the same contact structure on the boundary $\partial(C_n, J_n') = (\partial C_n, \xi')$ as on $\partial(B_n, J_n)$, thus we will show $\xi' \cong \xi \cong \xi_{std}$.

We label the unknots in the plumbing diagram of C_n, (as seen in Figure 15), $W_1, W_2, \ldots W_{n-1}$. We put a Stein structure J_n' on C_n, seen in Figure 16, by making the unknots, representing the spheres in the plumbing configuration, Legendrian in such a way that the framing of each unknot corresponds to the required framing as dictated by Theorem 3.5: $tb(W_i) - 1$. Observe, that in this particular choice of Legendrian representatives of unknots, we have $rot(W_1) = -n$, $rot(W_2) = \cdots = rot(W_{n-1}) = 0$.

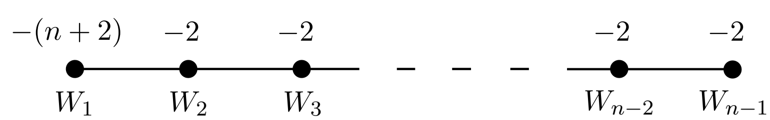


FIGURE 15. **Plumbing diagram of C_n, $n \geq 2$**

Let $\lambda_1, \ldots, \lambda_{n-1}$ be the meridians of the knots $W_1, \ldots, W_{n-1}$. Also, let $\beta_1, \ldots, \beta_{n-1}$ be the basis of $H_2(C_n; \mathbb{Z})$ determined by $W_1, \ldots, W_{n-1}$. The relations of $\lambda_1, \ldots, \lambda_{n-1}$ in $H_1(\partial C_n; \mathbb{Z})$ are as follows:

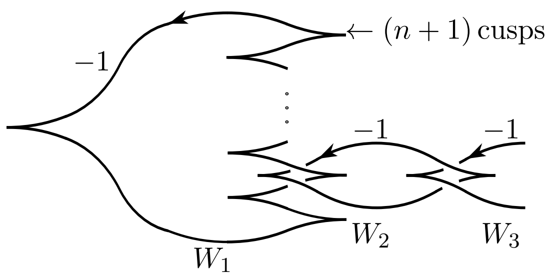


FIGURE 16. **Stein structure J_n' on C_n**

$$\left.\begin{array}{l} (-n-2)\lambda_1 + \lambda_2 = 0 \\ \lambda_1 - 2\lambda_2 + \lambda_3 = 0 \\ \lambda_2 - 2\lambda_3 + \lambda_4 = 0 \\ \vdots \\ \lambda_{n-3} - 2\lambda_{n-2} + \lambda_{n-1} = 0 \\ \lambda_{n-2} - 2\lambda_{n-1} = 0 \end{array}\right\} \Longrightarrow \begin{array}{l} \lambda_2 = (n+2)\lambda_1 \\ \lambda_3 = (2n+3)\lambda_1 \\ \lambda_4 = (3n+4)\lambda_1 \\ \vdots \\ \lambda_{n-1} = (n^2-n-1)\lambda_1 \\ (n^2)\lambda_1 = 0\,. \end{array}$$

As before, for n odd, $\partial C_n = L(n^2, n-1)$ has only one spin structure, $\mathfrak{r}$, represented by the characteristic sublink $L(\mathfrak{r}) = W_2 + W_4 + W_6 + \cdots + W_{n-1}$, in addition, we have $L_0 = \emptyset$. Again, letting ρ be as in Theorem 3.5, we have:

$$\begin{array}{rcl} \langle \rho, \beta_1 \rangle &=& \frac{1}{2}(rot(W_1) + \ell k(W_1, W_2 + W_4 + \cdots + W_{n-1})) = \frac{1}{2}(-n+1) = \frac{1-n}{2} \\ \langle \rho, \beta_2 \rangle &=& \frac{1}{2}(rot(W_2) + \ell k(W_2, W_2 + W_4 + \cdots + W_{n-1})) = \frac{1}{2}(0-2) = -1 \\ \langle \rho, \beta_3 \rangle &=& \frac{1}{2}(rot(W_3) + \ell k(W_3, W_2 + W_4 + \cdots + W_{n-1})) = \frac{1}{2}(0+2) = 1 \\ &\vdots& \\ \langle \rho, \beta_{n-2} \rangle &=& \frac{1}{2}(rot(W_{n-2}) + \ell k(W_{n-2}, W_2 + W_4 + \cdots + W_{n-1})) = \frac{1}{2}(0+2) = 1 \\ \langle \rho, \beta_{n-1} \rangle &=& \frac{1}{2}(rot(W_{n-1}) + \ell k(W_{n-1}, W_2 + W_4 + \cdots + W_{n-1})) = \frac{1}{2}(0-2) = -1\,. \end{array}$$

Using the above, we compute $PD\Gamma_{\partial C_n}(\xi', \mathfrak{t})$:

$$\begin{aligned}
PD\Gamma_{\partial C_n}(\xi', \mathfrak{t}) &= \langle \rho, \beta_1 \rangle \lambda_1 + \cdots + \langle \rho, \beta_{n-1} \rangle \lambda_{n-1} \\
&= \frac{1-n}{2}\lambda_1 - \lambda_2 + \lambda_3 - \cdots + \lambda_{n-2} - \lambda_{n-1} \\
&= \frac{1-n}{2}\lambda_1 - (n+2)\lambda_1 + (2n+3)\lambda_1 - \cdots - (n^2-n-1)\lambda_1 \\
&= \frac{1-n}{2}\lambda_1 - \frac{n^2+1}{2}\lambda_1 \\
&\equiv \frac{n^2-n}{2}\lambda_1 \mod n^2\,.
\end{aligned}$$

For n even, $\partial C_n = L(n^2, n-1)$ has two spin structures $\mathfrak{t}_1$ and $\mathfrak{t}_2$, corresponding to the characteristic sublinks $L(\mathfrak{t}_1) = W_1 + W_3 + W_5 + \cdots + W_{n-1}$ and $L(\mathfrak{t}_2) = \emptyset$ respectively, (and $L_0 = \emptyset$ as before). For the spin structure $\mathfrak{t}_1$, we have:

$$\begin{aligned}
\langle \rho, \beta_1 \rangle &= \frac{1}{2}(rot(W_1) + \ell k(W_1, W_1 + W_3 + \cdots + W_{n-1})) = -(n+1) \\
\langle \rho, \beta_2 \rangle &= \frac{1}{2}(rot(W_2) + \ell k(W_2, W_1 + W_3 + \cdots + W_{n-1})) = 1 \\
\langle \rho, \beta_3 \rangle &= \frac{1}{2}(rot(W_3) + \ell k(W_3, W_1 + W_3 + \cdots + W_{n-1})) = -1 \\
&\vdots \\
\langle \rho, \beta_{n-2} \rangle &= \frac{1}{2}(rot(W_{n-2}) + \ell k(W_{n-2}, W_1 + W_3 + \cdots + W_{n-1})) = 1 \\
\langle \rho, \beta_{n-1} \rangle &= \frac{1}{2}(rot(W_{n-1}) + \ell k(W_{n-1}, W_1 + W_3 + \cdots + W_{n-1})) = -1\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
PD\Gamma_{\partial C_n}(\xi', \mathfrak{t}_1) &= \langle \rho, \beta_1 \rangle \lambda_1 + \cdots + \langle \rho, \beta_{n-1} \rangle \lambda_{n-1} \\
&= -(n+1)\lambda_1 + \lambda_2 - \lambda_3 + \cdots + \lambda_{n-2} - \lambda_{n-1} \\
&= -(n+1)\lambda_1 + (n+2)\lambda_1 - (2n+3)\lambda_1 + \cdots - (n^2-n-1)\lambda_1 \\
&\equiv \frac{n^2-n}{2}\lambda_1 \mod n^2\,.
\end{aligned}$$

For the spin structure $\mathfrak{r}_2$ we get:

$$\begin{aligned}
\langle \rho, \beta_1 \rangle &= \frac{1}{2}(rot(W_1) + \ell k(W_1, \emptyset)) = \frac{-n}{2} \\
\langle \rho, \beta_2 \rangle &= \frac{1}{2}(rot(W_2) + \ell k(W_2, \emptyset)) = 0 \\
&\vdots \\
\langle \rho, \beta_{n-1} \rangle &= \frac{1}{2}(rot(W_{n-1}) + \ell k(W_{n-1}, \emptyset)) = 0\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
PD\Gamma_{\partial C_n}(\xi', \mathfrak{r}_2) &= \langle \rho, \beta_1 \rangle \lambda_1 + \cdots + \langle \rho, \beta_{n-1} \rangle \lambda_{n-1} \\
&= \frac{-n}{2}\lambda_1 \\
&\equiv \frac{2n^2 - n}{2}\lambda_1 \mod n^2\,.
\end{aligned}$$

3.3.4. *Showing* $(\partial B_n, \xi) \cong (L(n^2, n-1), \xi_{std}) \cong (\partial C_n, \xi')$.

Proposition 3.6. *With the notation established above, we have* $(\partial B_n, \xi) \cong (L(n^2, n-1), \xi_{std}) \cong (\partial C_n, \xi')$ *as contact 3-manifolds. In particular, this implies that* (B_n, ω_n) *is a symplectic filling of* $(L(n^2, n-1), \xi_{std})$.

Proof. Since we computed the Γ invariant for the manifolds $(\partial B_n, \xi)$, $(L(n^2, n-1), \xi_{std})$, and $(\partial C_n, \xi')$, in order to show these manifolds have the same contact structure, $(\xi \cong \xi_{std} \cong \xi')$, we have to find a suitable identification between these manifolds, in particular between their first homology groups. It is important to note that the contact structures ξ and ξ' are tight [Lis], since they were induced from the boundaries of Stein surfaces. Therefore, due to the classification of tight contact structures on lens spaces $L(p, q)$ [Gi] [Ho], the Γ invariant is sufficient to show the isomorphisms between these contact 3-manifolds. This is because the Γ invariant shows which *spin*^{c} structures are induced by the contact structures ξ, ξ', and ξ_{std}, since $\Gamma(\zeta, \cdot) : Spin(M) \to H_1(M; \mathbb{Z})$ depends only on the homotopy class $[\zeta]$.

Figure 17 demonstrates a sequence of Kirby calculus moves from ∂B_n^* to $-\partial\nu(C)$, (compare with Figure 11). (Note, for shorthand we represent most spheres by dots, as in Figure 1.) As the moves are performed, we keep track of the $\mu_i \in H_1(\partial B_n^*;\mathbb{Z})$, the meridians of the associated unknots in the diagram. In move **I** we perform n blow-ups. In moves **II** and **III** we perform a handleslide. In moves $\mathbf{IV}_1,\ldots,\mathbf{IV}_{n-3}$ we perform a handleslide in each. Finally, in move **V**, we blow-down the unknot with framing (-1).

As a result we can form the following identifications between $\mu_1,\mu_2 \in H_1(\partial B_n;\mathbb{Z})$ and $\nu_0,\ldots,\nu_{n+1} \in H_1(L(n^2,n-1);\mathbf{Z})$:

$$
\begin{aligned}
\mu_1 &= \nu_{n+1}\\
n\mu_1 + n\mu_2 &= \nu_n\\
(n-1)\mu_1 + (n-1)\mu_2 &= \nu_{n-1}\\
&\vdots\\
2\mu_1 + 2\mu_2 &= \nu_2\\
\mu_1 + \mu_2 &= \nu_1\,.
\end{aligned}
\tag{3.5}
$$

For n odd, $L(n^2,n-1)$ has only one spin structure, so there is no need to keep track of it throughout the Kirby moves. We multiply both sides of the first identification above by $\frac{n^2-n}{2}$ and get:

$$
\begin{aligned}
\frac{n^2-n}{2}\mu_1 &= \frac{n^2-n}{2}\nu_{n+1} = (n+1)\frac{n^2-n}{2}\nu_1 = \frac{n^3-n}{2}\nu_1\\
&\equiv \left(\frac{n^3-n}{2} - \frac{(n-1)n^2}{2}\right)\nu_1 \equiv \frac{n^2-n}{2}\nu_1 \mod n^2\,.
\end{aligned}
$$

Thus, we have:

$$
PD\Gamma_{(\partial B_n)}(\xi,\mathfrak{s}) = \frac{n^2-n}{2}\mu_1 \equiv \frac{n^2-n}{2}\nu_1 = PD\Gamma_{L(n^2,(n-1))}(\xi_{std},\mathfrak{t})\,.
$$

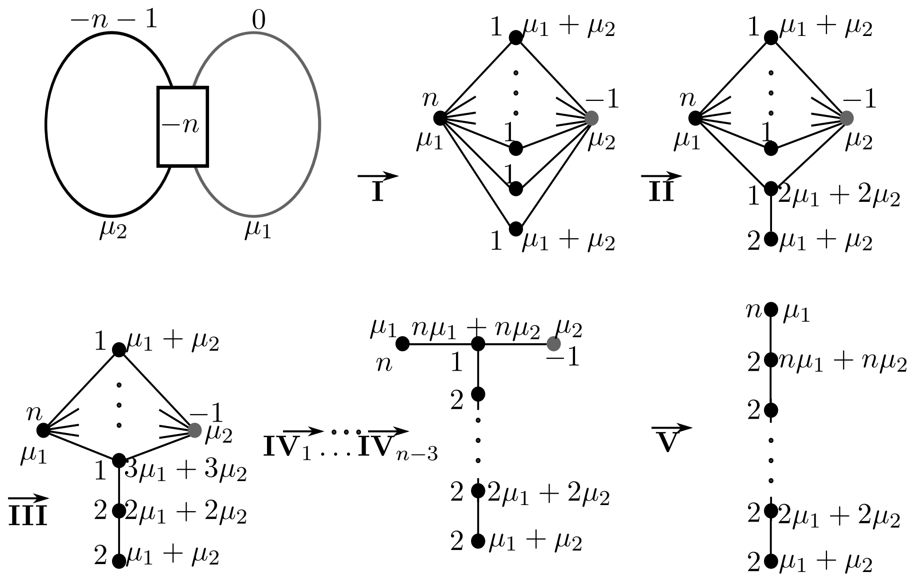


FIGURE 17. **Kirby moves from ∂B_n^* to $-\partial\nu(C)$**

For n even, since $L(n^2, n-1)$ has two spin structures, in addition to matching up the μ_i to the ν_i, we also also have to make an appropriate identification among the spin structures. In Figure 17 we follow the spin structure $\mathfrak{s}_1$ through the Kirby moves by denoting the knots corresponding to its characteristic sublink in grey color. Thus, we can see that spin structure $\mathfrak{s}_1$ of ∂B_n is identified with the spin structure $\mathfrak{t}_1$ of $-\partial\nu(C)$. If we multiply the first identification of (3.5) by $\frac{n^2-n}{2}$, we get:

$$\begin{aligned} \frac{n^2-n}{2}\mu_1 &= \frac{n^2-n}{2}\nu_{n+1} = (n+1)\frac{n^2-n}{2}\nu_1 = \frac{n^3-n}{2}\nu_1 \\ &\equiv \frac{2n^2-n}{2}\nu_1 \mod n^2 . \end{aligned}$$

Likewise, if we take the last identification of (3.5), and apply the relations for μ_i, we get $(1-n)\mu_1 = \nu_1$. We multiply this by $\frac{n^2-n}{2}$, and get:

$$\frac{n^2-n}{2}\nu_1 = \frac{n^2-n}{2}(1-n)\mu_1 \equiv \frac{2n^2-n}{2}\mu_1 \mod n^2 .$$

As a result, we have:

$$PD\Gamma_{(\partial B_n)}(\xi, \mathfrak{s}_1) = \frac{2n^2-n}{2}\mu_1 \equiv \frac{n^2-n}{2}\nu_1 = PD\Gamma_{L(n^2,(n-1))}(\xi_{std}, \mathfrak{t}_1) \tag{3.6}$$

$$PD\Gamma_{(\partial B_n)}(\xi, \mathfrak{s}_2) = \frac{n^2-n}{2}\mu_1 \equiv \frac{2n^2-n}{2}\nu_1 = PD\Gamma_{L(n^2,(n-1))}(\xi_{std}, \mathfrak{t}_2)\,. \tag{3.7}$$

As a consequence, this gives us $(\partial B_n, \xi) \cong (L(n^2, n-1), \xi_{std})$.

In a similar manner, we can show $(\partial B_n, \xi) \cong (\partial C_n, \xi')$. We first find a suitable identification between $\mu_1, \mu_2 \in H_1(\partial B_n; \mathbb{Z})$ and $\lambda_1, \ldots, \lambda_{n-1} \in H_1(\partial C_n; \mathbb{Z})$, by a sequence of Kirby moves depicted in Figure 18. In move **I** we perform a handleslide: we slide K_1^n over K_2^n. In moves **II** and **III** we perform blow-ups. In moves $\mathbf{IV}_1, \ldots, \mathbf{IV}_{n-4}$ we perform a blow-up in each. Finally, in move **V** we blow-down the unknot with framing (1).

In the final diagram we can see a clear identification with Figure 15, which gives us the following:

$$\begin{aligned} \mu_1 + \mu_2 &= \lambda_1 \\ 2\mu_1 + \mu_2 &= \lambda_2 \\ &\vdots \\ (n-2)\mu_1 + \mu_2 &= \lambda_{n-2} \\ (n-1)\mu_1 + \mu_2 &= \lambda_{n-1}\,. \end{aligned} \tag{3.8}$$

For n odd, the above identifications imply:

$$\frac{n^2-n}{2}\mu_1 \equiv \frac{n^2-n}{2}\lambda_1 \mod n^2\,. \tag{3.9}$$

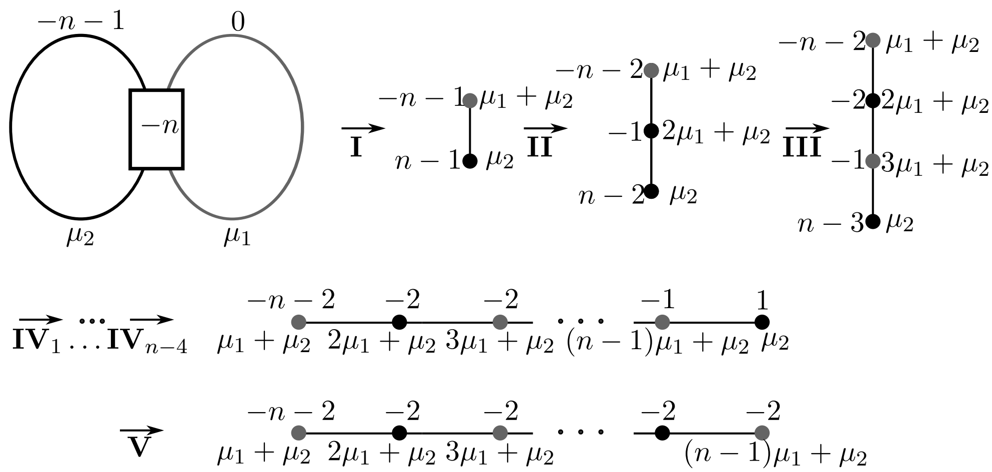


FIGURE 18. **Kirby moves from ∂B_n to ∂C_n**

Giving us:

$$PD\Gamma_{(\partial B_n)}(\xi, \mathfrak{s}) \equiv PD\Gamma_{\partial C_n}(\xi', \mathfrak{r})\,. \tag{3.10}$$

For n even, we again match up the spin structures by following the spin structure $\mathfrak{s}_1$ through the Kirby moves in Figure 18. We represent the spin structure $\mathfrak{s}_1$ by coloring the corresponding unknots in its characteristic sublink with a grey color. Thus, we can see that the spin structure $\mathfrak{s}_1$ of ∂B_n is identified with the spin structure $\mathfrak{r}_1$ of ∂C_n. Similar to previous calculations, the relations in (3.8) imply:

$$\frac{2n^2 - n}{2}\mu_1 \equiv \frac{n^2 - n}{2}\lambda_1 \mod n^2 \tag{3.11}$$

$$\frac{n^2 - n}{2}\mu_1 \equiv \frac{2n^2 - n}{2}\lambda_1 \mod n^2\,. \tag{3.12}$$

Therefore,

$$PD\Gamma_{(\partial B_n)}(\xi, \mathfrak{s}_1) \equiv PD\Gamma_{\partial C_n}(\xi', \mathfrak{r}_1) \tag{3.13}$$

$$PD\Gamma_{(\partial B_n)}(\xi, \mathfrak{s}_1) \equiv PD\Gamma_{\partial C_n}(\xi', \mathfrak{r}_1)\,. \tag{3.14}$$

$\square$

3.4. **Symplectic rational blow-up - main theorem.**

3.4.1. *Lagrangian cores $\mathcal{L}_{n,q}$.* In this section we define the symplectic rational blow-up operation. It is important to note, that just like the symplectic blow-up is not unique because of the choice of radius of the removed 4-ball, so to, the symplectic rational blow-up will also not be unique due to the choice of the symplectic volume of the removed rational homology ball B_n. Moreover, we also have to make a choice of the symplectic structure on the B_ns. Therefore, we will go further, and show that the existence of a certain 2-dimensional Lagrangian core (see section 3.1) in a symplectic manifold (X, ω) will have a standard neighborhood that will be our desired symplectic rational homology ball (B_n, ω_n) as in section 3.3.2.

Now we will describe the construction of our Lagrangian cores. First, we take an embedding $\gamma : S^1 \hookrightarrow (X, \omega)$. Next, we consider a Lagrangian immersion $\mathcal{L} : D^2 \looparrowright (X, \omega)$, (an embedding on the interior of D), such that its boundary "wraps around" $\gamma(S^1)$, with winding number n, so $\gamma(S^1) \hookrightarrow \mathcal{L}(\partial D)$. There is another winding number q that comes in to the picture, so we are going to call this Lagrangian disk immersion $\mathcal{L}_{n,q}$. Let P be the following bundle over $\gamma(S^1)$:

$$P = \bigcup_{z \in \gamma(S^1)} \{\text{plane } \pi | \pi \subset T_z X, \text{ oriented}, \omega(\pi) = 0, T_z(\gamma(S^1)) \subset \pi\}.$$

Because we are restricting to those planes π that contain $T_z(\gamma(S^1))$, the bundle P is an S^1-bundle. So, after a choice of trivialization, we have $P \cong S^1 \times \gamma(S^1)$, and a map:

$$\begin{aligned} \widehat{\mathcal{L}_{n,q}} &: \partial D \to P \cong S^1 \times \gamma(S^1) \\ \widehat{\mathcal{L}_{n,q}} &: x \longmapsto (\mathcal{L}_{n,q})_*(T_x D) \end{aligned} \tag{3.15}$$

where n is the degree of the map $\widehat{\mathcal{L}_{n,q}}$ on the first component, and q on the second. Note, that before a choice of trivialization of P, q is only defined mod n.

Now we state the formal definition of the Lagrangian "cores", $\mathcal{L}_{n,q}$:

Definition 3.7. Let $\mathcal{L}_{n,q} : D \looparrowright (X, \omega)$ be a smooth Lagrangian immersion of a 2-disk D into a symplectic 4-manifold (X, ω), with $n \geq 2$ an integer, and q is an integer defined mod n, assuming the following conditions:

i) $\mathcal{L}_{n,q}(D - \partial D) \hookrightarrow (X, \omega)$ is a smooth embedding.

ii) There exists a smooth embedding $\gamma : S^1 \hookrightarrow (X, \omega)$ such that $\gamma(S^1) \hookrightarrow \mathcal{L}_{n,q}(\partial D)$.

iii) The pair (n, q) are defined to be the degrees of the maps on the first and second component, respectively of the map $\widehat{\mathcal{L}_{n,q}} : \partial D \to P \cong S^1 \times \gamma(S^1)$ as defined in (3.15).

iv) The map $\widehat{\mathcal{L}_{n,q}}$ is injective, so for any points $x, y \in \partial D$ if $\mathcal{L}_{n,q}(x) = \mathcal{L}_{n,q}(y)$ then $(\mathcal{L}_{n,q})_*(T_x(D)) \neq (\mathcal{L}_{n,q})_*(T_y(D))$.

Figure 9 is an illustration of how $\mathcal{L}_{n,q}(D)$ looks like near $\gamma(S^1)$, for $n = 3$ and $q = 1$. Note, we will use $\mathcal{L}_{n,q}$ to also denote its image in (X, ω).

3.4.2. *Statement of the main theorem.* Now we are ready to state the main theorem:

Theorem 3.8. ***Symplectic Rational Blow-Up.*** *Suppose $\mathcal{L}_{n,1} \subset (X, \omega)$, is as in Definition 3.7 with $q = 1$, then for some small $\lambda > 0$, there exists a symplectic embedding of $(B_n, \lambda\omega_n)$ in (X, ω), and for some $\lambda_0 < \lambda$ and $\mu > 0$, there exists a symplectic 4-manifold (X', ω') such that $(X', \omega') = ((X, \omega) - (B_n, \lambda_0\omega_n)) \cup_\phi (C_n, \mu\omega'_n)$, where ϕ is a symplectic map, and (B_n, ω_n) and (C_n, ω'_n) are the symplectic manifolds as defined in section 3.3. (X', ω') is called the* ***symplectic rational blow-up*** *of (X, ω).*

Proof. The proof of the theorem will follow from Lemmas 3.9 and 3.10 below, but first we will introduce some notation.

We express $\mathcal{L}_{n,q} \subset (X, \omega)$ as a union:

$$\mathcal{L}_{n,q} = \Sigma_{n,q} \cup \Delta, \tag{3.16}$$

where $\Sigma_{n,q}$ is the image of a collar neighborhood of $\partial D \subset D$, C_D, and Δ is the image of the remainder $D - C_D$. First, we will present a model of $\Sigma_{n,q}$ explicitly by expressing it in terms of local coordinates.

For $\mathcal{L}_{n,q}$, the respective $\gamma(S^1) \hookrightarrow (X, \omega)$, as in Definition 3.7, will have a neighborhood, $S^1 \times D^3$ with standard Darboux coordinates: (θ, x, u, v) with the symplectic form $\omega = d\theta \wedge dx + du \wedge dv$, where θ is a 2π-periodic coordinate on S^1, and x, u, v are the standard coordinates on D^3. Parameterizing C_D by (t, s) with $0 \leq t < 2\pi$ and $0 \leq s \leq \epsilon$ for some small ϵ, Definition 3.7 implies that without loss of generality, $\Sigma_{n,q}(t, s)$ can be expressed as:

$$\Sigma_{n,q}(t, s) = (nt, x(t, s), s\cos(\psi_q(t, s)), -s\sin(\psi_q(t, s))) \tag{3.17}$$

where $x(t, s)$ and $\psi_q(t, s)$ are smooth functions with $x(0, s) = x(2\pi, s)$ and $\psi_q(2\pi, s) - \psi_q(0, s) = q(2\pi)$. We observe that at $s = 0$ we have:

$$\Sigma_{n,q}(t, 0) = (nt, 0, 0, 0) = \gamma(S^1)\,.$$

Thus, the numbers in the pair (n, q) as they appear in (3.17), are the degrees of the maps in part (iii) of Definition 3.7.

Next, we switch to somewhat more convenient coordinates (θ, x, τ, ρ), (sometimes referred to as action-angle coordinates) where:

$$\theta \to \theta, \quad x \to x, \quad u \to \sqrt{2\rho}\cos\tau, \quad v \to -\sqrt{2\rho}\sin\tau\,.$$

This coordinate change is symplectic, since the symplectic form remains the same: $\omega = d\theta \wedge dx + d\tau \wedge d\rho$. We can reparameterize $\Sigma_{n,q}$ with (t, I), $0 \leq t < 2\pi$ and

$0 \leq I \leq \epsilon'$, where $I = \frac{1}{2}s^2$, and so (3.17) in (θ, x, τ, ρ) coordinates becomes:

$$\Sigma_{n,q}(t, I) = (nt, x(t, I), \psi_q(t, I), I)\,.$$

The Lagrangian condition $\omega_{|T_{\mathcal{L}_{n,q}(D)}X} = 0$ imposes further restrictions on $x(t, I)$, thus $\Sigma_{n,q}(t, I)$ can be given as follows:

$$\Sigma_{n,q}(t, I) = (nt, -\frac{q}{n} I \frac{\partial \psi_q}{\partial t} + \int \frac{q}{n} I \frac{\partial^2 \psi_q}{\partial I \partial t}\, dI, \psi_q(t, I), I)\,. \tag{3.18}$$

A particular example is when $\psi_q(t, I) = qt$, this will be called $\Sigma^\sharp_{n,q}$:

$$\Sigma^\sharp_{n,q}(t, I) = (nt, -\frac{q}{n} I, qt, I)\,. \tag{3.19}$$

Again, we refer the reader to Figure 9 for an illustration of $\Sigma^\sharp_{n,q}$ for $n = 3$ and $q = 1$.

Lemma 3.9. *Let $\mathcal{L}_{n,q} \subset (X, \omega)$ be as in Definition 3.7. Then there exists another $\mathcal{L}^\sharp_{n,q} \subset (X, \omega)$, also as in Definition 3.7, such that if $\mathcal{L}_{n,q} = \Sigma_{n,q} \cup \Delta$, (as defined in (3.16)), then $\mathcal{L}^\sharp_{n,q} = \Sigma^\sharp_{n,q} \cup \Delta^\sharp$, where $\Sigma^\sharp_{n,q}$ is as in (3.19) and $\Delta^\sharp$ agrees with Δ everywhere except for a small neighborhood of its boundary. We will refer to such $\mathcal{L}^\sharp_{n,q}$s as the "good" ones. Thus all the "good" $\mathcal{L}_{n,q}$s are the ones which are standard in a neighborhood of $\gamma(S^1)$.*

Lemma 3.10. *Let $\mathcal{L}^\sharp_{n,q}$ and $\check{\mathcal{L}}^\sharp_{n,q}$ be both "good" $\mathcal{L}_{n,q}$s, in accordance with Definition 3.7 and Lemma 3.9, then they will have symplectomorphic neighborhoods in (X, ω).*

Note, the above Lemmas are meant to mirror the standard Weinstein Lagrangian embedding theorem. First, we will prove Lemma 3.9 by constructing a Hamiltonian vector flow that will take $\Sigma_{n,q}$ to $\Sigma^\sharp_{n,q}$. Second, we will prove Lemma 3.10 using Lemma 3.9 and a relative Moser type argument.

Proof. Proof of Lemma 3.9. We construct a Hamiltonian H with flow

$$\varphi_\alpha : nbhd(\widetilde{\gamma(S^1)}) \to nbhd(\widetilde{\gamma(S^1)}),$$

for $0 \leq \alpha \leq 1$, where $\widetilde{\gamma(S^1)}$ is the n-sheeted covering space of $\gamma(S^1)$. Note, we choose ϵ' small enough such that $\widetilde{\Sigma^\sharp_{n,q}}(t, I) \subset nbhd(\widetilde{\gamma(S^1)})$. H and φ_α are as given in (3.20) and (3.21) below on $\widetilde{\Sigma^\sharp_{n,q}}(t, I)$ and are 0 otherwise:

$$\varphi_\alpha(\theta, x, \tau, \rho) = (\theta, x - (\frac{\partial f}{\partial\theta}\rho - \int \frac{\partial^2 f}{\partial\rho\partial\theta}\rho\, d\rho)\alpha, \tau + f(\theta, \rho)\alpha, \rho) \tag{3.20}$$

$$H(\theta, x, \tau, \rho) = \int f(\theta, \rho)\, d\rho \tag{3.21}$$

for some continuous function f.

The following calculations show that φ_α preserves the symplectic form $\omega = d\theta \wedge dx + d\tau \wedge d\rho$, and that it is indeed the Hamiltonian flow for the H above.

$$\begin{aligned}
& d\theta \wedge d(x - (\frac{\partial f}{\partial\theta}\rho - \int \frac{\partial^2 f}{\partial\rho\partial\theta}\rho\, d\rho)\alpha) + d(\tau + f(\theta,\rho)\alpha) \wedge d\rho \\
= \; & d\theta \wedge (dx - \alpha(\frac{\partial^2 f}{\partial\theta^2}\rho d\theta + \frac{\partial f}{\partial\theta}d\rho + \frac{\partial^2 f}{\partial\rho\partial\theta}\rho d\rho - \frac{\partial}{\partial\theta}(\int \frac{\partial^2 f}{\partial\rho\partial\theta}\rho\, d\rho)d\theta - \frac{\partial^2 f}{\partial\rho\partial\theta}\rho d\rho)) \\
+ \; & (d\tau + \alpha(\frac{\partial f}{\partial\theta}d\theta + \frac{f}{\partial\rho}d\rho)) \wedge d\rho \\
= \; & d\theta \wedge dx - \alpha\frac{\partial f}{\partial\theta}d\theta \wedge d\rho + d\tau \wedge d\rho + \alpha\frac{\partial f}{\partial\theta}d\theta \wedge d\rho \\
= \; & d\theta \wedge dx + d\tau \wedge d\rho\,.
\end{aligned}$$

Also,

$$\frac{d}{d\alpha}\varphi_\alpha = (0, -\frac{\partial f}{\partial\theta}\rho + \int \frac{\partial^2 f}{\partial\rho\partial\theta}\rho\, d\rho, f(\theta,\rho), 0) = (\frac{\partial H}{\partial x}, -\frac{\partial H}{\partial\theta}, \frac{\partial H}{\partial\rho}, -\frac{\partial H}{\partial\tau})\,.$$

If we let $p_n : nbhd(\widetilde{\gamma(S^1)}) \to nbhd(\gamma(S^1))$ be the $(n:1)$ covering map, then we have $p_n \circ \varphi_1(\widetilde{\Sigma^\sharp_{n,q}}) = \Sigma_{n,q}$, taking $f(nt, I) = \psi_q(t, I) - qt$, as seen in the equation below:

$$\begin{aligned} p_n \circ \varphi_1(\widetilde{\Sigma^\sharp_{n,q}})(t,I) &= (nt, -\frac{q}{n}I - \frac{\partial f(nt,I)}{\partial(nt)}I + \int \frac{\partial^2 f(nt,I)}{\partial I \partial(nt)} I\, dI, t + f(nt,I), I) \\ &= (nt, -\frac{q}{n}I\frac{\partial \psi_q}{\partial t} + \int \frac{q}{n} I \frac{\partial^2 \psi_q}{\partial I \partial t}\, dI, \psi_q(t,I), I) \\ &= \Sigma_{n,q}(t,I)\,. \end{aligned}$$

Note, in order for $p_n \circ \varphi_\alpha(\widetilde{\Sigma^\sharp_{n,q}})$ to remain being a "$\Sigma_{n,q}$" for all $0 \leq \alpha \leq 1$, (and not "tear" as α goes from 0 to 1), we must have

$$[q(2\pi) + (\psi_q(2\pi, I) - q(2\pi)\alpha] - [q(0) + (\psi_q(I,0) - q(0))\alpha]$$

be an integer multiple of 2π for all $0 \leq \alpha \leq 1$. This implies:

$$\psi_q(2\pi, I) - \psi_q(0, I) = q(2\pi)\,.$$

Which is precisely the condition that $\psi_q(t,I)$ needs to have in the definition of $\Sigma_{n,q}(t,I)$. Hence, whenever we have $\mathcal{L}_{n,q} \subset (X,\omega)$, we can always find a "good" $\mathcal{L}^\sharp_{n,q} \subset (X,\omega)$, which looks "standard" near $\gamma(S^1)$, by the map $p_n \circ \varphi_1^{-1}(\widetilde{\Sigma_{n,q}}) = \Sigma^\sharp_{n,q}$, with $\mathcal{L}^\sharp_{n,q} = \Sigma^\sharp_{n,q} \cup \Delta^\sharp$. (We have $\Delta^\sharp$, since the map $p_n \circ \varphi_1^{-1}$ gets smoothed off near $\partial\Delta$.) □

Proof. Proof of Lemma 3.10. In order to prove this lemma, we will be using the relative Moser's theorem, stated below:

Lemma 3.11. ***Relative Moser's Theorem.*** [EM] *Let ω_t be a family of symplectic forms on a compact manifold W with full-dimensional submanifold W_1, such that $\omega_t = \omega_0$ over an open neighborhood of W_1 and the relative cohomology class $[\omega_t - \omega_0] \in H^2(W, W_1)$ vanishes for all $t \in [0,1]$. Then there exists an isotopy $\Phi_t : W \to W$ which is fixed on an open neighborhood of W_1 and such that $\Phi_t^*(\omega_0) = \omega_t$, $t \in [0,1]$.*

(Note, in [EM] this thereom is stated for the pair $(W, \partial W)$, however, the proof directly extends to the pair (W, W_1).)

Let $\mathcal{L}^{\sharp}_{n,q}$ be a "good" $\mathcal{L}_{n,q}$ immersed disk, and let $\mathcal{L}^{0,\sharp}_{n,q} \hookrightarrow (X_0, \omega_0)$ be some particular "good" $\mathcal{L}_{n,q}$ immersed disk in a symplectic 4-manifold (X_0, ω_0). Let $\Sigma^{\sharp,\delta}_{n,q}(t, I) \subset \Sigma^{\sharp}_{n,q}(t, I)$ be such that $0 \leq t < 2\pi$ and $\delta \leq I < \epsilon'$. Then, we let

$$\begin{aligned} \mathcal{L}^{\sharp,\delta}_{n,q} &= \Sigma^{\sharp,\delta}_{n,q}(t, I) \cup \Delta^{\sharp} \\ \mathcal{L}^{0,\sharp,\delta}_{n,q} &= \Sigma^{\sharp,\delta}_{n,q}(t, I) \cup \overset{\circ}{\Delta}{}^{\sharp} . \end{aligned}$$

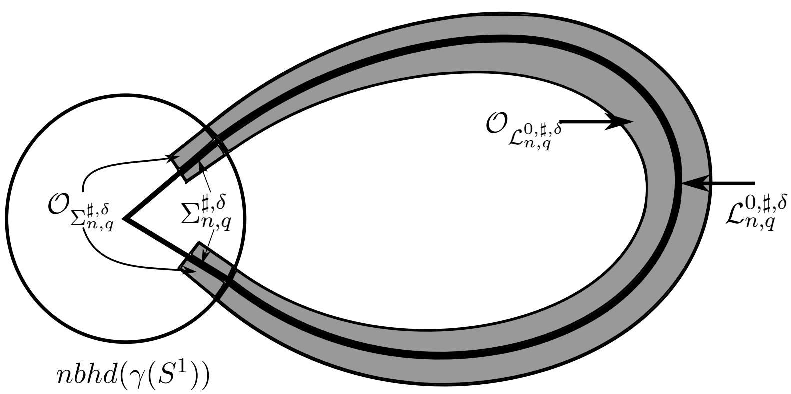


FIGURE 19. **Schematic diagram of $\mathcal{O}_{\mathcal{L}^{0,\sharp,\delta}_{n,q}}$**

Also, let $\nu(X, \mathcal{L}^{\sharp,\delta}_{n,q})$ and $\nu(X_0, \mathcal{L}^{0,\sharp,\delta}_{n,q})$ be normal bundles of $\mathcal{L}^{\sharp,\delta}_{n,q}$ and $\mathcal{L}^{0,\sharp,\delta}_{n,q}$ respectively. We also denote

$$\begin{aligned} N_{\Sigma^{\sharp,\delta}_{n,q}} \subset N_{\mathcal{L}^{\sharp,\delta}_{n,q}} &\subset \nu(X, \mathcal{L}^{\sharp,\delta}_{n,q}) \\ \mathcal{O}_{\Sigma^{\sharp,\delta}_{n,q}} \subset \mathcal{O}_{\mathcal{L}^{0,\sharp,\delta}_{n,q}} &\subset \nu(X_0, \mathcal{L}^{0,\sharp,\delta}_{n,q}) \end{aligned}$$

to be the neighborhoods of $\Sigma^{\sharp,\delta}_{n,q}$, $\mathcal{L}^{\sharp,\delta}_{n,q}$ and $\mathcal{L}^{0,\sharp,\delta}_{n,q}$ in their respective normal bundles. Refer to Figure 19 for a schematic diagram. We construct a bundle map:

$$B_0 : T_x(\nu(X_0, \mathcal{L}^{0,\sharp,\delta}_{n,q})) \longrightarrow T_y(\nu(X, \mathcal{L}^{\sharp,\delta}_{n,q}))$$

for $x \in \mathcal{L}^{0,\sharp,\delta}_{n,q}$ and $y \in \mathcal{L}^{\sharp,\delta}_{n,q}$ such that $B_0|_{\Sigma^{\sharp,\delta}_{n,q}} = Id$. By the Whitney Extension theorem [Wh], we have a map

$$\phi_0 : \mathcal{O}_{\mathcal{L}^{0,\sharp,\delta}_{n,q}} \to N_{\mathcal{L}^{\sharp,\delta}_{n,q}}$$

with $\phi = B_0$ on $T_{\mathcal{L}^{0,\sharp,\delta}_{n,q}}(\nu(X_0, \mathcal{L}^{0,\sharp,\delta}_{n,q}))$ and $\phi_0^*(\omega) = \omega_0$ on $\mathcal{O}_{\Sigma^{\sharp,\delta}_{n,q}}$.

Next, we define a family of symplectic forms:

$$\omega_t = (1-t)\omega_0 + t\phi_0^*(\omega) \text{ for } t \in [0,1] \,.$$

We get $\omega_t - \omega_0 = t(\phi_0^*(\omega) - \omega_0) = 0$, for all $t \in [0,1]$ on some open neighborhood of $\mathcal{O}_{\Sigma^{\sharp,\delta}_{n,q}}$. We can do this by making our ϵ' a bit smaller. Moreover, we can pass down to the relative homology class:

$$[\omega_t - \omega_0] \equiv [t(\phi_o^*(\omega) - \omega_0)] \in H^2(\mathcal{O}_{\mathcal{L}^{0,\sharp,\delta}_{n,q}}, \mathcal{O}_{\Sigma^{\sharp,\delta}_{n,q}}) \,.$$

This relative class $[\omega_t - \omega_0]$ will vanish since $\phi_0^*(\omega) = \omega_0$ on $\mathcal{O}_{\Sigma^{\sharp,\delta}_{n,q}}$. Thus, we can use relative Moser's theorem (Lemma 3.11), with $W = \mathcal{O}_{\mathcal{L}^{0,\sharp,\delta}_{n,q}}$ and $W_1 = \mathcal{O}_{\Sigma^{\sharp,\delta}_{n,q}}$, and we get an isotopy $\Phi_t : \mathcal{O}_{\mathcal{L}^{0,\sharp,\delta}_{n,q}} \to \mathcal{O}_{\mathcal{L}^{0,\sharp,\delta}_{n,q}}$ such that $\Phi_1^*(\omega_0) = \omega_1 = \phi_0^*(\omega)$. We define the map $\Phi_\sharp = \phi_0 \circ \Phi_1^{-1}$, and obtain:

$$\Phi_\sharp : \mathcal{O}_{\mathcal{L}^{0,\sharp,\delta}_{n,q}} \to N_{\mathcal{L}^{\sharp,\delta}_{n,q}} \text{ with } \Phi_\sharp^*(\omega) = \omega_0 \,.$$

Likewise, we can obtain a symplectomorphism $\check{\Phi}_\sharp : N_{\check{\mathcal{L}}^{\sharp,\delta}_{n,q}} \to \mathcal{O}_{\mathcal{L}^{0,\sharp,\delta}_{n,q}}$. By composing $\Phi_\sharp$ and $\check{\Phi}_\sharp$, we get a symplectomorphism:

$$\Phi : N_{\check{\mathcal{L}}^{\sharp,\delta}_{n,q}} \to N_{\mathcal{L}^{\sharp,\delta}_{n,q}} \,,$$

which extends to map between $\check{\mathcal{L}}^{\sharp}_{n,q}$ and $\mathcal{L}^{\sharp}_{n,q}$, since they are both "good" immersed disks, and are the same on $\Sigma^{\sharp}_{n,q}$.

Now to complete the proof of Lemma 3.10, we will construct a particular model of a neighborhood of such an immersed Lagrangian disk $\mathcal{L}^{0,\sharp}_{n,q} = \Sigma^{\sharp}_{n,q}(t,I) \cup \overset{\circ}{\Delta}$. We will do this by symplectically gluing $N_{\Sigma^{\sharp}_{n,q}}$ to $N_B \subset T^*(B)$, where $T^*(B)$ is just the cotangent space of a 2-disk B, and N_B is its neighborhood in $T^*(B)$. With the identification of $\Sigma^{\sharp,\delta}_{n,q}$ with C_B, a collar neighborhood of the boundary of disk B, we can construct a symplectomorphism Ψ between $N_{\Sigma^{\sharp,\delta}_{n,q}} \subset \nu(X, \mathcal{L}^{\sharp,\delta}_{n,q})$ and $N_{C_B} \subset T^*(B)$, by a similar Moser type argument as used above. We then symplectically glue $N_{\Sigma^{\sharp}_{n,q}}$ to N_B via Ψ. $\square$

3.4.3. *Showing* $(nbhd\, \mathcal{L}^{\sharp}_{n,1}) \cong (B_n, \omega_n)$. Now that we have shown that a neighborhood of a "good" Lagrangian core $\mathcal{L}^{\sharp}_{n,1}$ is entirely standard, we will now show that this standard neighborhood is in fact equivalent to (B_n, ω_n) for each $n \geq 2$, where ω_n are the symplectic forms induced on the rational homology balls B_n by the Stein structures J_n, in section 3.3.2. Note, there is a choice in the size of a neighborhood of $\mathcal{L}^{\sharp}_{n,1}$ which corresponds to the choice of the symplectic volume of the rational homology ball B_n; this is the source of the non-uniqueness of the symplectic rational blow-up operation, as mentioned in section 3.4.1.

Lemma 3.12. *There exists a neighborhood of* $\mathcal{L}^{\sharp}_{n,1}$ *in* (X, ω), $N(\mathcal{L}^{\sharp}_{n,1})$, *such that there exists a symplectomorphism*

$$f : (N(\mathcal{L}^{\sharp}_{n,1}), \omega|_{N(\mathcal{L}^{\sharp}_{n,1})})^+ \to (B_n, \omega_n)^+ \tag{3.22}$$

where $(N(\mathcal{L}^{\sharp}_{n,1}), \omega|_{N(\mathcal{L}^{\sharp}_{n,1})})^+$ *and* $(B_n, \omega_n)^+$ *are the symplectic completions of* $(N(\mathcal{L}^{\sharp}_{n,1}),$ $\omega|_{N(\mathcal{L}^{\sharp}_{n,1})})$ *and* (B_n, ω_n) *respectively, as defined in section 2.2.*

Proof. Recall that the "good" Lagrangian cores $\mathcal{L}^{\sharp}_{n,1}$ can be expressed as a union $\mathcal{L}^{\sharp}_{n,1} = \Sigma^{\sharp}_{n,1}(t,I) \cup \Delta^{\sharp}$, and that $\Sigma^{\sharp,\delta}_{n,q}(t,I) \subset \Sigma^{\sharp}_{n,q}(t,I)$ is such that $0 \leq t < 2\pi$ and

$\delta \leq I < \epsilon'$. We fix a number $0 < a < \epsilon'$ and let:

$$\partial(\Sigma^{\sharp}_{n,1} - \Sigma^{\sharp,a}_{n,1}) = \mathcal{K}_{n,1} \tag{3.23}$$

where $\mathcal{K}_{n,1}$ is a knot in $\partial(S^1 \times D^3) \cong S^1 \times S^2$, and the spheres S^2 have radius a. The knot $\mathcal{K}_{n,1}$ can be described with respect to the (θ, x, τ, ρ) coordinates, introduced in section 3.4.2, as follows:

$$\mathcal{K}_{n,1}(t) = (nt, -\frac{a}{n}, t, a)\,. \tag{3.24}$$

We observe that $\mathcal{K}_{n,1}$ is a Legendrian knot with respect to the standard (tight) contact structure on $S^1 \times S^2$, which has the contact 1-form

$$\alpha = -xd\theta - \rho d\tau \tag{3.25}$$

with the restriction to the spheres $x^2 + 2\rho = a^2$.

In light of Eliashberg's classification of Stein handlebodies [El2] and Gompf's result [Go2], seen here in Theorem 3.3, in order to show that a neighborhood of $\mathcal{L}^{\sharp}_{n,1}$ is the same symplectic manifold as (B_n, ω_n), then all we have to show is that $\mathcal{K}_{n,1}$ in (3.23) is the same Legendrian knot as K_2^n in Figure 13. We will show this by presenting the knot $\mathcal{K}_{n,1}$ in $S^1 \times S^2$ in an alternate way, and showing that this is equivalent to the presentation of the knot K_2^n in *standard form* as in Figure 13.

In ([Go2], section 2) Gompf presents an alternate way of presenting a knot in $S^1 \times S^2$, we recreate this method here. We want to pull back the contact 1-form $\alpha = -xd\theta - \rho d\tau$ to $\mathbb{R}^3$ using cylindrical coordinates (θ, r, ϖ), by stereographically projecting all of the spheres S^2, (with radius a), in $S^1 \times S^2$. Thus, when we perform the stereographic projections, we switch from coordinate system (θ, x, τ, ρ) to (θ, r, ϖ), such that:

$$\begin{aligned}
\theta &= \theta \\
x &= \frac{a(r^2-1)}{r^2+1} \\
\tau &= -\varpi \\
\rho &= \frac{2a^2r^2}{(r+1)^2}\,.
\end{aligned}$$

Consequently, the contact 1-form $\alpha = -xd\theta - \rho d\tau$ restricted to the spheres $x^2+2\rho = a^2$, becomes the following contact 1-form on $S^1 \times (S^2 - \{poles\})$:

$$\tilde{\alpha} = d\varpi + \frac{1-r^4}{2ar^2}d\theta\,,$$

which after rescaling pulls back to standard contact 1-form on $\mathbb{R}^3$,

$$\alpha_{std} = dZ + XdY$$

(with the Z coordinate being 2π-periodic). As a result, we can present knots in $S^1 \times S^2$ by their standard *front* projections into the Y-Z plane, i.e. by projecting them to the θ-ϖ "plane" $\mathbb{R}^2/2\pi\mathbb{Z}^2$. Thus, one can alternately present knots in $S^1 \times S^2$ by disconnected arcs in a square, corresponding to $\mathbb{R}^2/2\pi\mathbb{Z}^2$.

Now we will present the knot $\mathcal{K}_{n,1}$, using this alternate presentation. First, we transfer the knot $\mathcal{K}_{n,1}$ into (θ, r, ϖ) coordinates,

$$\tilde{\mathcal{K}}_{n,1} = (nt, C_{a,n}, -t)\,, \tag{3.26}$$

where $C_{a,n}$ is just a constant depending on a and n. Figure 20 depicts the *front* projection of $\tilde{\mathcal{K}}_{n,1}$ onto the θ-ϖ plane, (after we shift it in the θ-coordinate, and take $-\frac{\pi}{2} \leq t \leq \frac{3\pi}{2}$). We then perform Gompf's move 6 (see [Go2], Figure 11), which in effect swings the knot around the 1-handle, and we obtain the knot as seen in Figure 21, which is isotopic to the knot K_2^n in *standard form* in Figure 13.

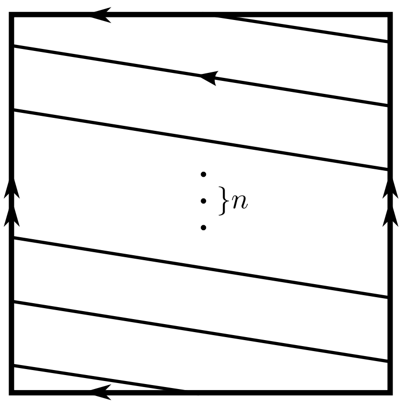


FIGURE 20.

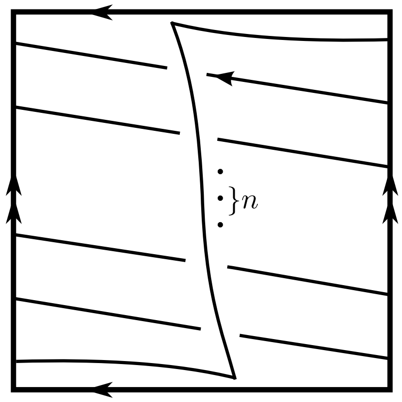


FIGURE 21.

Remark 3.13. To see how to compute the classical Legendrian knot invariants from a diagram like in Figure 20, we describe what happens to the rotation number. For a Legendrian knot K in a contact 3-manifold, and v a nonvanishing vector field in the contact planes, one can define the rotation number $rot_v(K) = rot(K)$, as the signed number of times the tangent vector field of K rotates, relative to v, in the contact planes [Go2]. This number is independent of the choice of the nonvanishing vector field v. In the presentations of knots in $S^1 \times S^2$, by their front projections in $\mathbb{R}^2/2\pi\mathbb{Z}^2$ (and knots in standard form), we can choose v to be $\frac{\partial}{\partial X}$ inside the square (or box). This corresponds to computing $rot(K)$ with counting cusps, as in (3.2). However, when we extend the vector field $\frac{\partial}{\partial X}$ to a nonvanishing vector field on all of $S^1 \times S^2$, then the latter vector field will make a $360°$ twist going from the top edge of the square, $\mathbb{R}^2/2\pi\mathbb{Z}^2$, to the bottom. Consequently, one can compute the rotation number of a Legendrian knot in $\mathbb{R}^2/2\pi\mathbb{Z}^2$ by counting the cusps as in equation (3.2) and adding to that $\pm$ the number of times the knot crosses over from the top to the bottom edge of the square.

As a result, both $(N(\mathcal{L}^{\sharp}_{n,1}), \omega|_{N(\mathcal{L}^{\sharp}_{n,1})})$ and (B_n, ω_n) can be represented by the same Kirby-Stein diagram, i.e. Figure 13. Thus, there exists a symplectomorphism between the symplectic completions of these two manifolds. $\square$

Lemma 3.12 implies that for a small enough λ, ($\lambda << 1$), we can find a symplectomorphic copy of (B_n, ω_n) in (X, ω) as follows: let ι be the identification of the copy of $(N(\mathcal{L}^{\sharp}_{n,1}), \omega|_{N(\mathcal{L}^{\sharp}_{n,1})})$ in $(N(\mathcal{L}^{\sharp}_{n,1}), \omega|_{N(\mathcal{L}^{\sharp}_{n,1})})^{+}$ to the copy of $(N(\mathcal{L}^{\sharp}_{n,1}), \omega|_{N(\mathcal{L}^{\sharp}_{n,1})})$ in (X, ω), then we have an embedding:

$$\iota \circ f^{-1}(B_n, \lambda\omega_n) \hookrightarrow (X, \omega) \tag{3.27}$$

where f is the symplectomorphism in (3.22). As a consequence, combining the results of Lemmas 3.9, 3.10 and 3.12, we have shown that for each $n \geq 2$, if there exists a Lagrangian core $\mathcal{L}_{n,1} \subset (X, \omega)$, then for a small enough λ, there exists an embedding of the rational homology ball: $(B_n, \lambda\omega_n) \hookrightarrow (X, \omega)$; hence proving the first part of Theorem 3.8. Note, as stated before, just like the symlectic blow-up, the symplectic rational blow-up operation is unique up to the choice of volume of the rational homology ball B_n, i.e. the choice of a λ that works for this construction.

3.4.4. *Gluing argument using computations of Gompf's invariant.* In the final step of our proof of Theorem 3.8, we will show using Proposition 3.6, that we can symplectically rationally blow-up (X, ω) by removing $(B_n, \lambda_0\omega_n)$ and replacing it with $(C_n, \mu\omega'_n)$, for some $\lambda_0 < \lambda$ and $\mu > 0$.

We start by assuming that we have $\mathcal{L}_{n,1} \subset (X, \omega)$, implying that we can find an embedding $(B_n, \lambda\omega_n) \hookrightarrow (X, \omega)$. According to Proposition 3.6, $\partial(B_n, \omega_n) \cong (L(n^2, n-1), \xi_{std}) \cong \partial(C_n, \omega'_n)$, thus for some high enough t we will have a symplectomorphism:

$$g : [t, \infty) \times \partial(B_n, \omega_n) \to [t, \infty) \times \partial(C_n, \omega'_n) \tag{3.28}$$

such that

$$\begin{aligned} [t, \infty) \times \partial(B_n, \omega_n) \quad &\subset \quad (B_n, \omega_n)^{+} \\ [t, \infty) \times \partial(C_n, \omega'_n) \quad &\subset \quad (C_n, \omega'_n)^{+} \end{aligned}$$

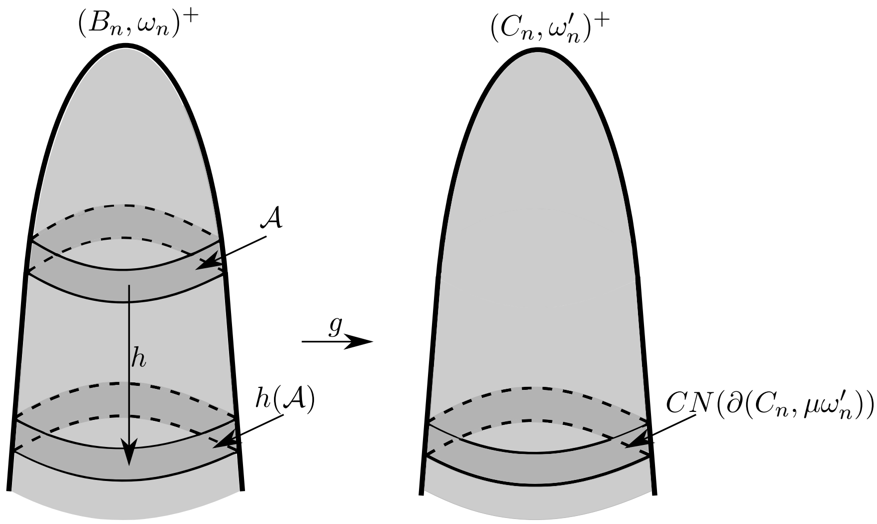


FIGURE 22. Symplectic completions of (B_n, ω_n) and (C_n, ω_n')

where $(B_n, \omega_n)^+$ and $(C_n, \omega_n')^+$ are the symplectic completions of (B_n, ω_n) and (C_n, ω_n') respectively. We take the embedding $(B_n, \lambda\omega_n) \hookrightarrow (X, \omega)$, and consider its image $f \circ \iota^{-1}(B_n, \lambda\omega_n)$ back in $(B_n, \omega_n)^+$. Likewise, for $\lambda_0 < \lambda$, we can consider the image of $f \circ \iota^{-1}(B_n, \lambda_0\omega_n)$ in $(B_n, \omega_n)^+$. We define the $\mathcal{A} \subset (B_n, \omega_n)^+$ to be:

$$\mathcal{A} = (f \circ \iota^{-1}(B_n, \lambda\omega_n)) - (f \circ \iota^{-1}(B_n, \lambda_0\omega_n)) \tag{3.29}$$

so that $\mathcal{A}$ is a collar neighborhood of the boundary of $f \circ \iota^{-1}(B_n, \lambda\omega_n)$.

We let h be the symplectomorphism corresponding to a radial vector field flow in $(B_n, \omega_n)^+$, then we can find a $\mu > 0$ such that $\mathcal{A} \subset (B_n, \omega_n)^+$ is symplectomorphic to $g \circ h(\mathcal{A}) \cong CN(\partial(C_n, \mu\omega_n')) \subset (C_n, \omega_n')^+$, where $CN(\partial(C_n, \mu\omega_n'))$ denotes a collar neighborhood of $\partial(C_n, \mu\omega_n')$ in $(C_n, \omega_n')^+$ (see Figure 22).

Finally, we are ready to construct the *symplectic rational blow-up* (X', ω') of (X, ω) (see Figure 23). We let:

$$(X', \omega') = ((X, \omega) - (B_n, \lambda_0\omega_n)) \cup_\phi (C_n, \mu\omega_n') \tag{3.30}$$

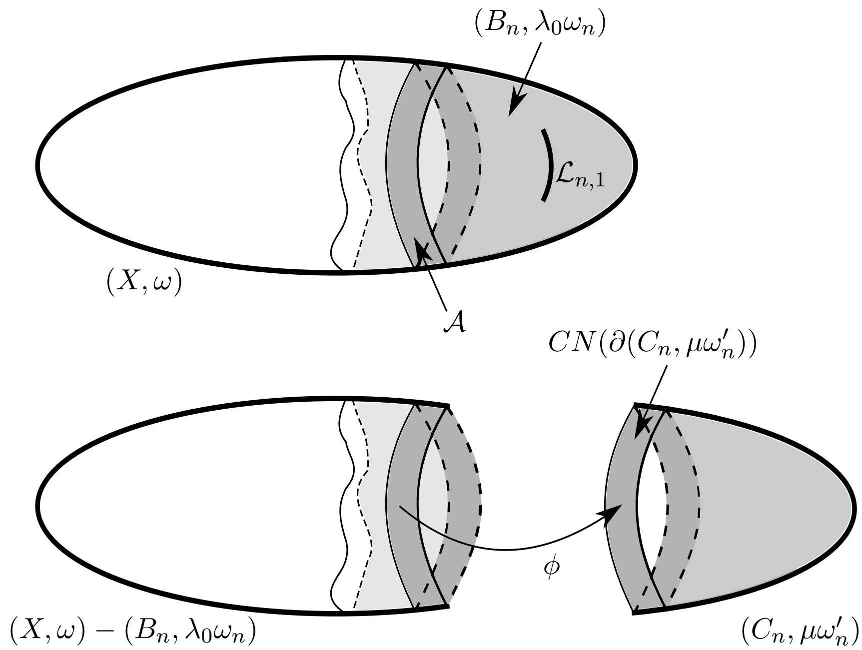


FIGURE 23. Construction on (X', ω')

where ϕ is the symplectic map:

$$\phi : \iota \circ f^{-1}(\mathcal{A}) \to CN(\partial(C_n, \mu\omega_n')).$$

□

It is worthwhile to note, that given the definition of the *symplectic rational blow-up*, one can ask the following symplectic capacity question: Given λ_0, what is the upper bound on μ such that the construction in (3.30) works?

4. Smooth embeddings of rational homology balls B_n

Now that we have defined the *symplectic rational blow-up*, we can ask which symplectic 4-manifolds can be rationally blown up? In other words, what are the obstructions to embedding a rational homology ball B_n in a symplectic 4-manifold? Given the definition of the *symplectic rational blow-up* in the previous section, this is equivalent to finding a Lagrangian core $\mathcal{L}_{n,1}$ (Definition 3.7) in a symplectic 4-manifold (X, ω). Moreover, we can ask, given a symplectic 4-manifold (X, ω), is there a bound on n for which one can embed B_n (see Conjecture 1.2)?

Before we study the obstructions to symplectic embeddings of rational homology balls B_n, we will first present a couple of theorems showing that on the smooth level, there is little obstruction to embedding the rational homology balls B_n.

The obvious examples of smooth 4-manifolds containing smoothly embedded rational homology balls B_n, are those manifolds obtained via a rational blow-down. Examples of such manifolds first appeared in Fintushel and Stern's original paper [FS2] on rational blow-downs: logarithmic transforms $E(m)_n$ of elliptic surfaces $E(m)$. In these manifolds, one starts with a fishtail fiber of $E(m)$, which has homological self-intersection 0, blows it up $(n-2)$ times, and then one obtains a configuration of spheres C_n, which one rationally blows down (see Figure 24). Consequently, one obtains a manifold $E(m)_n$, having the same (c_1^2, c_2) numbers but different Seiberg-Witten invariants as $E(m)$, which contains an embedded rational homology ball B_n (for SW invariants of $E(m)_n$ see Example 5.39).

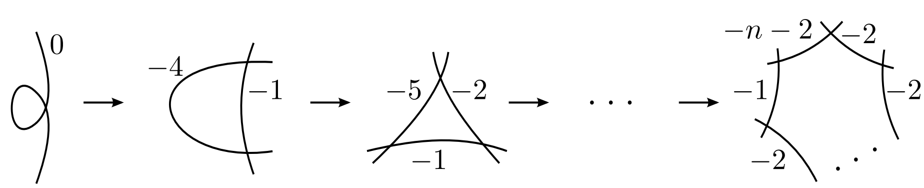


FIGURE 24. Fishtail fiber in $E(k)$ blown up $(n-1)$ times

Most other examples of smooth 4-manifolds that contain embedded rational homology balls B_n, are obtained in a similar manner, one blows up a smooth manifold several times, then finds a particular configuration of spheres C_n which one rationally blows down. Often, one ends up with a manifold with lower betti number b_2^- than the original manifold one started with. In fact, in a lot of these examples, since one can compute the betti numbers of the resultant manifold, by Freedman's theorem [Fr, FQ] one can conclude they are homeomorphic to $k\mathbb{C}P^2\#\ell\overline{\mathbb{C}P^2}$, for some k and ℓ. However, after a computation of the Seiberg-Witten invariants, one can often show that the resultant manifolds are not diffeomorphic to $k\mathbb{C}P^2\#\ell\overline{\mathbb{C}P^2}$, and thus possess an exotic smooth structure, which is frequently the goal. In fact, one can sometimes find an infinite family of exotic 4-manifolds which are homeomorphic but not diffeomorphic to $k\mathbb{C}P^2\#\ell\overline{\mathbb{C}P^2}$. For example, using these techniques, exotic $\mathbb{C}P^2\#7\overline{\mathbb{C}P^2}$ manifolds were constructed in [Pa2]. Additionally, using a generalized rational blowdown [Pa1] (also see section 6), exotic $\mathbb{C}P^2\#6\overline{\mathbb{C}P^2}$ manifolds were constructed in [SS].

4.1. **Main theorems on smooth embeddings of B_n.** The following two theorems give rise to a large class of smooth 4-manifolds which contain a smoothly embedded rational homology ball B_n:

Theorem 4.1. *Let V_{-n-1} be a neighborhood of a sphere with self-intersection number $(-n-1)$. There exists an embedding of the rational homology balls $B_n \hookrightarrow V_{-n-1}$, for all $n \geq 2$.*

Theorem 4.2. *Let V_{-4} be a neighborhood of a sphere with self-intersection number (-4). For all $n \geq 3$ odd, there exists an embedding of the rational homology balls $B_n \hookrightarrow V_{-4}$. For all $n \geq 2$ even, there exists an embedding of the rational homology balls $B_n \hookrightarrow B_2\#\overline{\mathbb{C}P^2}$.*

Proof. Proof of Theorem 4.1. We prove this using Kirby calculus (see section 2.1). We start with the Kirby diagram for V_{-n-1}, Figure 27, blow it up $(n-1)$ times, and obtain the configuration of spheres C_n with an additional sphere Σ_{-1} with $[\Sigma_{-1}]^2 = -1$,

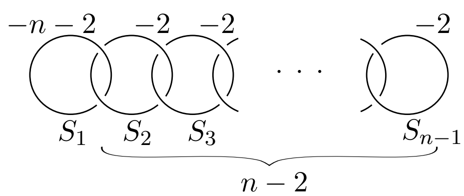


FIGURE 25.

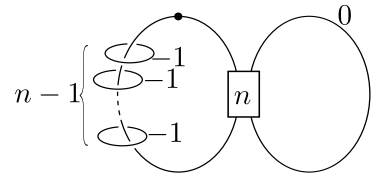


FIGURE 26.

attached to the last sphere with self-intersection (-2), S_{n-1}, Figure 30. In Figures 31-35 we proceed to do the standard Kirby calculus manipulation where one changes the Kirby diagram of C_n from the one in Figure 25 to the one in Figure 26, (see [GS], p. 516), by first adding a cancelling 1/2 handle pair (Figure 31) and performing a series of handleslides. However, in our case the additional sphere Σ_{-1} is present, and intersects with the C_n configuration in a non-trivial way. As a result, when we perform the last handleslide to get C_n to look like Figure 26, the sphere Σ_{-1} intersects with C_n as seen in Figure 35.

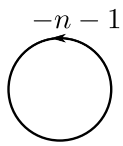


FIGURE 27. V_{-n-1}

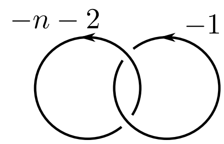


FIGURE 28. $V_{-n-1}\#\overline{\mathbb{C}P^2}$

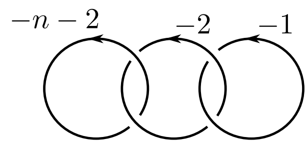


FIGURE 29. $V_{-n-1}\#2\overline{\mathbb{C}P^2}$

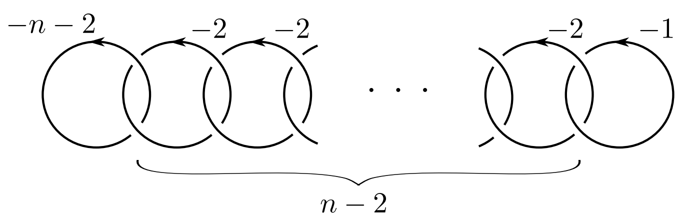


FIGURE 30. $V_{-n-1}\#(n-1)\overline{\mathbb{C}P^2}$

Next, in Figure 36 we perform the rational blow-down, thus replacing C_n with B_n. This is done by first swapping the one-handle and the 0-framed two-handle and then blowing down the $(n-1)$ spheres with self-intersection (-1). Consequently, after rationally blowing down, we obtain B_n with an additional 0-framed two-handle.

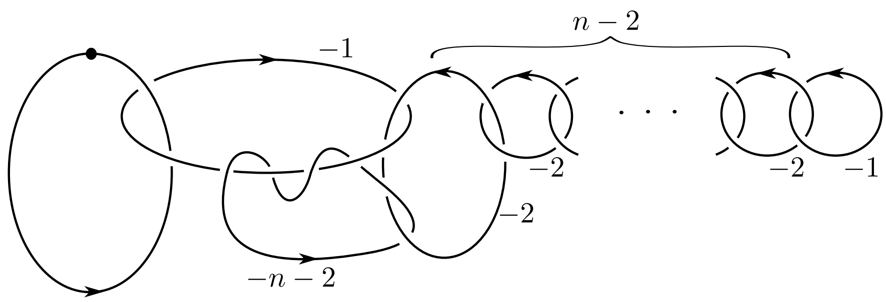


FIGURE 31.

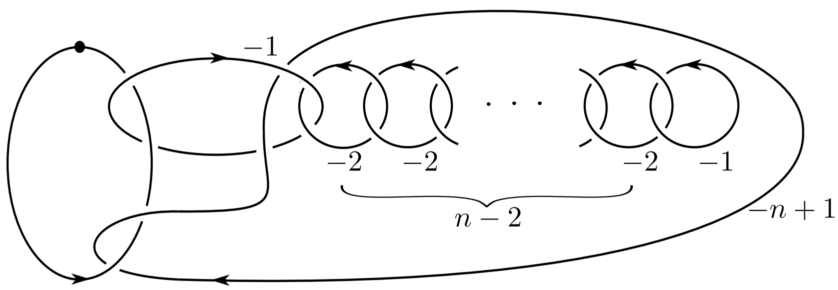
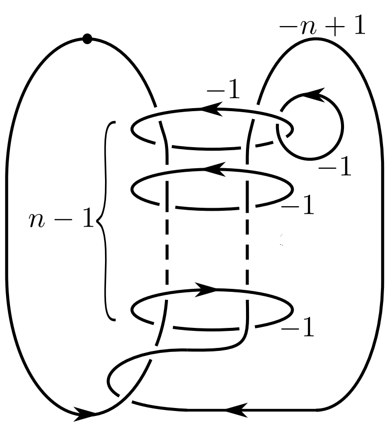


FIGURE 32.



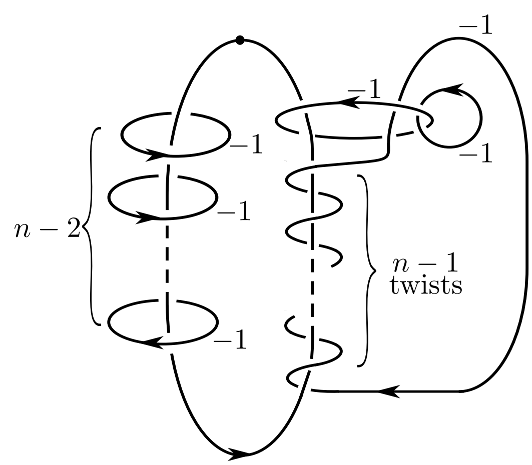


FIGURE 33. FIGURE 34.

When we slide then $(n-1)$-framed two-handle of B_n over that 0-framed two-handle, we obtain Figure 37. We proceed to slide the same handle over the 0-framed two-handle $(n-2)$ more times, and obtain Figure 38. Finally, we remove the cancelling

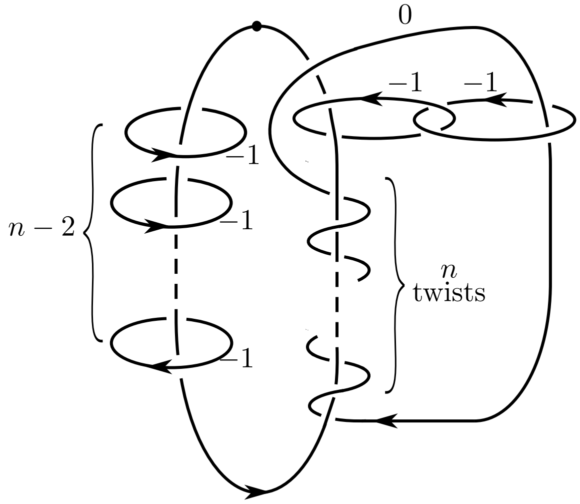


FIGURE 35.

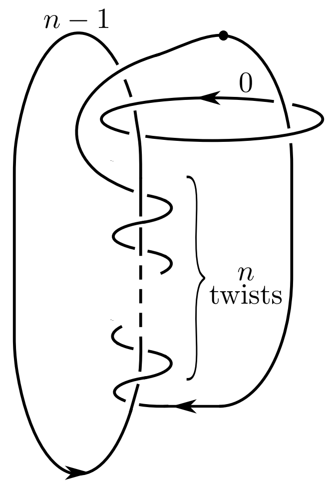


FIGURE 36. B_n with a two-handle

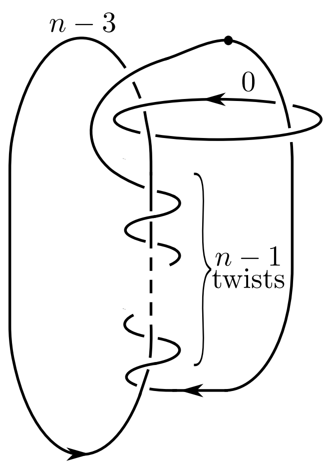


FIGURE 37.

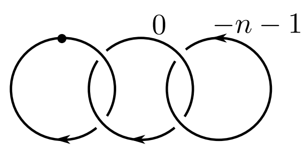


FIGURE 38.

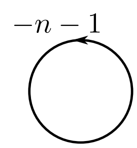


FIGURE 39. V_{-n-1}

1/2 handle pair, and obtain a single $(-n-1)$-framed two-handle, Figure 39, which is the manifold V_{-n-1}. Consequently, since to get from Figure 36 to Figure 39, we only performed handleslides, it follows that $B_n \hookrightarrow V_{-n-1}$. □

Corollary 4.3. *For $n \geq 2$, the rational blow-up of $B_n \hookrightarrow V_{-n-1}$ is diffeomorphic to $V_{-n-1}\#(n-1)\overline{\mathbb{C}P^2}$.*

In fact, the proof of Theorem 4.1, also proves Corollary 4.3. If we follow the Kirby moves backwards from Figure 39 to Figure 30, it follows that if we start with a V_{-n-1}, and rationally blow up the $B_n \hookrightarrow V_{-n-1}$, then we end up with $V_{-n-1}\#(n-1)\overline{\mathbb{C}P^2}$.

Proof. Proof of Theorem 4.2.

We prove Theorem 4.2 using similar Kirby Calculus techniques as in the proof of Theorem 4.1. (Note, the case $n = 2$ is trivial and the case $n = 3$ is covered in Theorem 4.1, so here we can assume $n \geq 4$.) We start with the Kirby diagram for V_{-4}, Figure 40. We blow up V_{-4} $(n-1)$ times, as seen in Figures 41 through 44, in such a manner that we end up with a plumbing tree of spheres as seen in Figure 44. This configuration of spheres is C_n with an extra sphere Σ'_{-1}, with self-intersection (-1), which intersects only with the first sphere with self-intersection (-2), S_2, (compare with Figure 30).

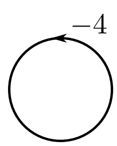


FIGURE 40. V_{-4}

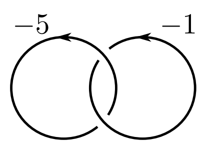


FIGURE 41. $V_{-4}\#\overline{\mathbb{C}P^2}$

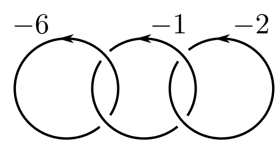


FIGURE 42. $V_{-4}\#2\overline{\mathbb{C}P^2}$

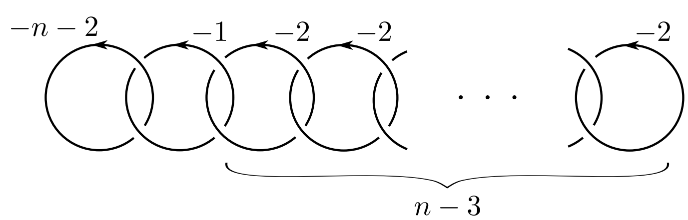


FIGURE 43. $V_{-4}\#(n-2)\overline{\mathbb{C}P^2}$

As was done in the proof of the previous theorem, we proceed with a series of Kirby moves that will change the presentation of C_n in Figure 44, from a linear plumbing of spheres, Figure 25, to the one in Figure 26. We start by adding a cancelling 1/2-handle pair in Figure 45. We proceed by sliding the $(-n-2)$-framed two-handle over the two-handle which was added in the previous step (Figure 46). Following this,

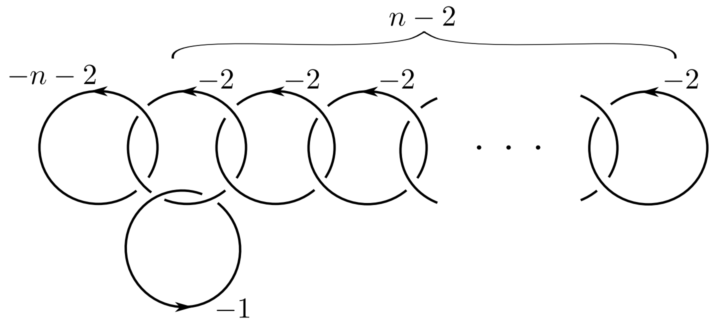


FIGURE 44. $V_{-4}\#(n-1)\overline{\mathbb{C}P^2}$

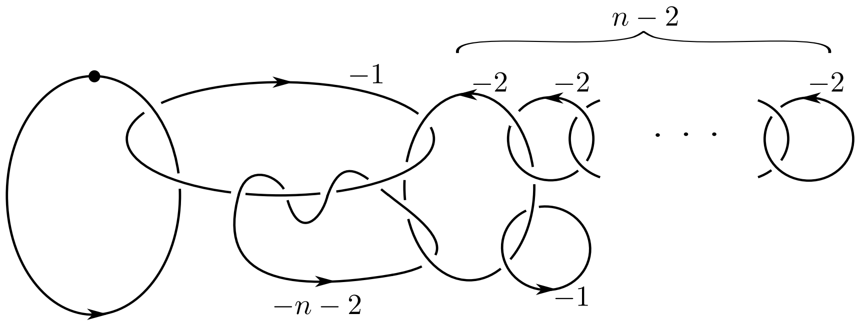


FIGURE 45.

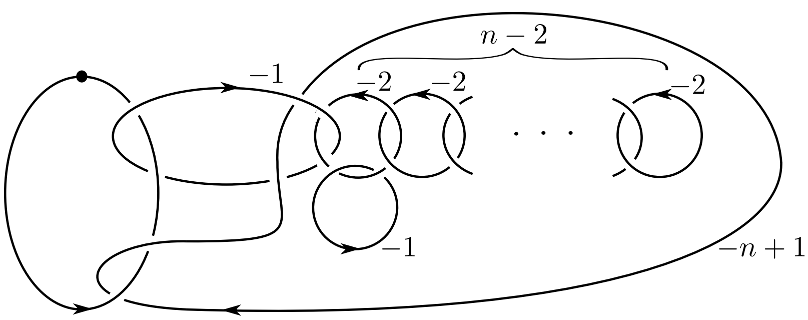


FIGURE 46.

we perform $(n-2)$ handleslides in order to slide off the (-2)-framed two-handles in Figures 47-49. As a result, the (-1)-framed two-handle corresponding to the sphere Σ'_{-1}, intersects once with each of the spheres corresponding to the $(n-2)$ (-1)-framed two-handles, as seen in Figure 49. Next, we slide the $(-n+1)$-framed two-handle off of each of the $(n-1)$ (-1)-framed two-handles, Figures 50 and 51. Consequently, in Figure 51 we obtain a presentation of C_n as in Figure 26, with the extra sphere Σ'_{-1}.

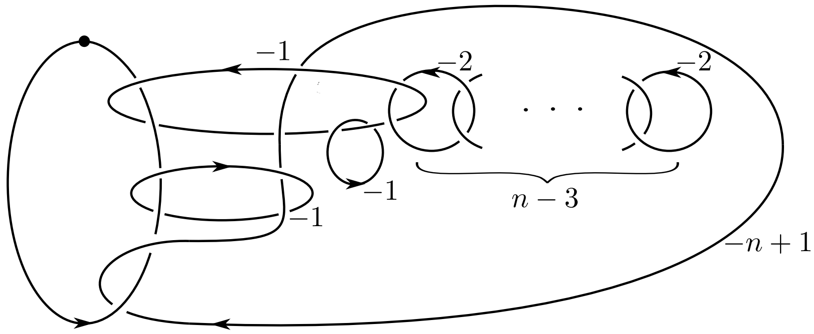


FIGURE 47.

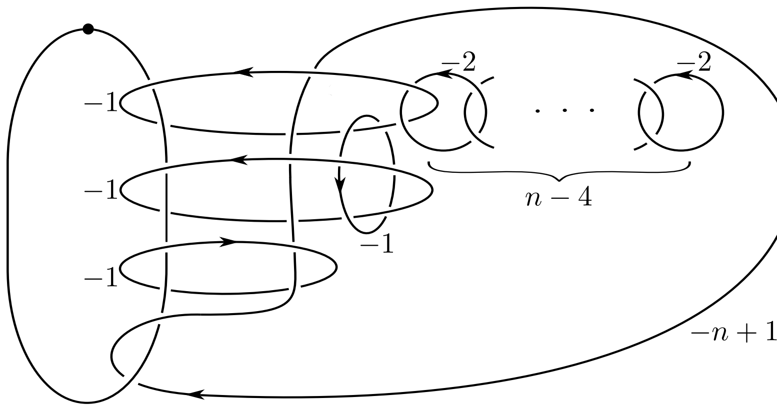
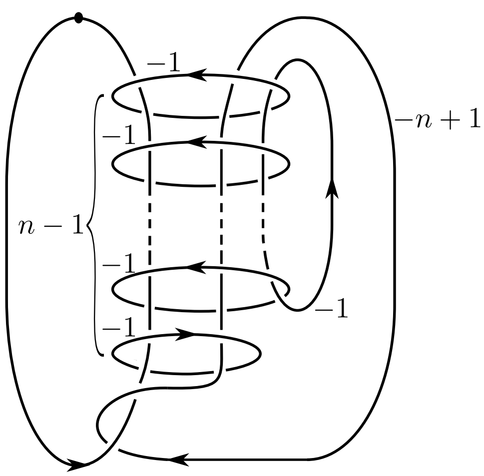


FIGURE 48.



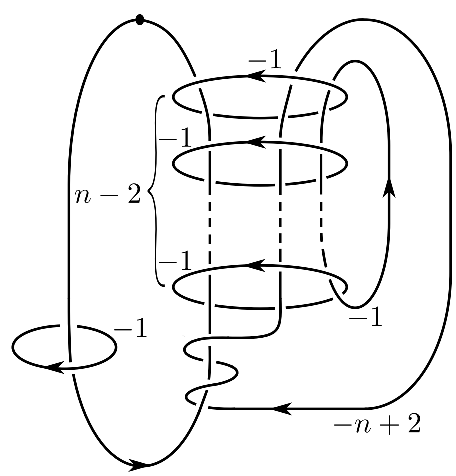


FIGURE 49. FIGURE 50.

Next, we perform the rational blow-down procedure, by exchanging the one-handle and the 0-framed two-handle, and blowing down along the $(n-1)$ spheres with self-intersection (-1), and obtain the Kirby diagram of B_n with an additional $(n-3)$-framed two-handle, Figure 52. Next, we slide the $(n-1)$-framed two-handle over

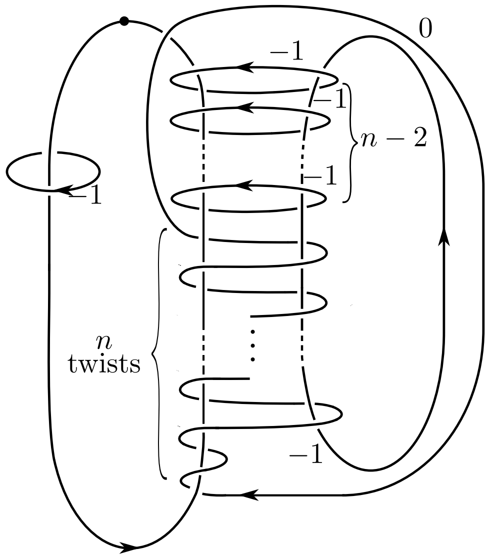


FIGURE 51.

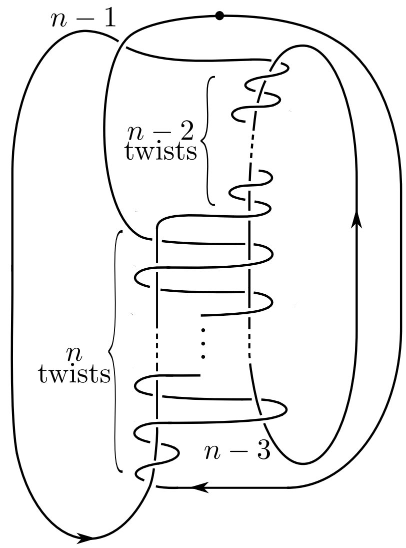


FIGURE 52. B_n with a two-handle

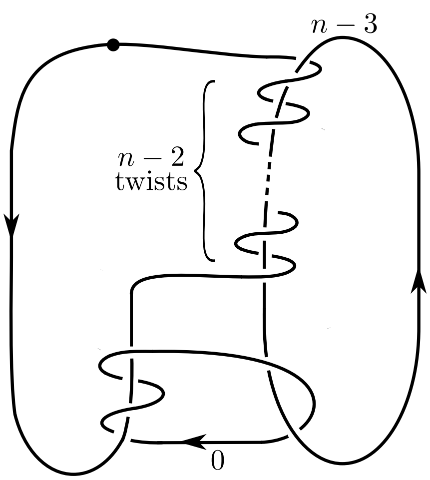


FIGURE 53.

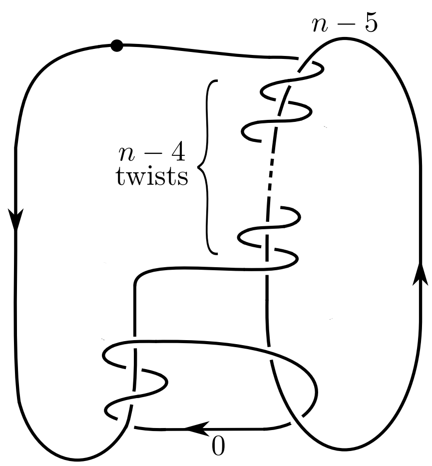


FIGURE 54.

the $(n-3)$-framed two-handle and obtain the Kirby diagram in Figure 53, with the $(n-1)$-framed two-handle becoming a 0-framed two-handle. At this point, the unknot corresponding to the $(n-3)$-framed two-handle is linked with the unknot corresponding to the one-handle with $(n-2)$ twists. If we slide off that $(n-3)$-framed two-handle off of the the 0-framed two-handle, then we knock down the amount of twists that the unknot corresponding to the $(n-3)$-framed two-handle is linked with the unknot corresponding to the one-handle by 2, thus obtaining Figure 54.

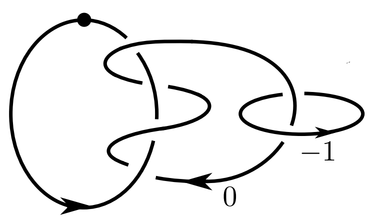


FIGURE 55.

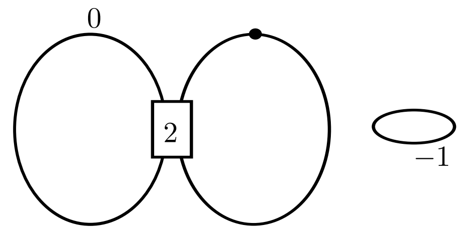


FIGURE 56. $B_2\#\overline{\mathbb{C}P^2}$

If n is even, then after $\frac{n-2}{2}$ such handleslides we obtain the diagram in Figure 55, (equivalent to the one in Figure 56), which is just B_2 blown up once, i.e. $B_2\#\overline{\mathbb{C}P^2}$. Consequently, if we start with $B_2\#\overline{\mathbb{C}P^2}$, and follow the Kirby moves backwards from Figure 56 to Figure 52, then we see that $B_n \hookrightarrow B_2\#\overline{\mathbb{C}P^2}$, for n even.

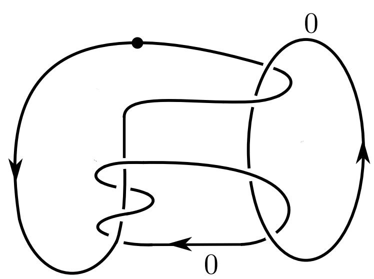


FIGURE 57.

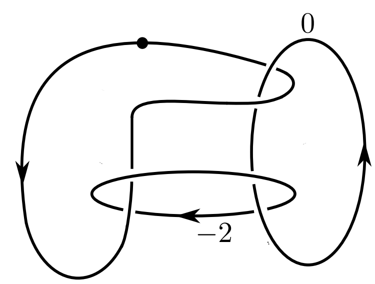


FIGURE 58.

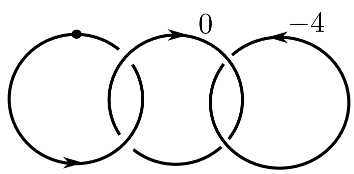


FIGURE 59.

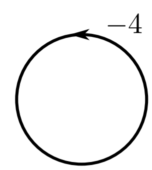


FIGURE 60. V_{-4}

If n is odd, then if we start with the diagram in Figure 53, and slide off the $(n-3)$-framed two-handle $\frac{n-3}{2}$ times, we obtain the diagram in Figure 57. Following this, we slide the 0-framed two handle (the one on the bottom of the diagram), over the other 0-framed two-handle and obtain the diagram in Figure 58. We then perform another handleslide, and slide off the (-2)-framed two-handle off of the 0-framed two-handle and get the diagram in Figure 59. Finally, we remove the cancelling 1/2-handle pair and are left with one (-4)-framed two-handle, Figure 60, which represents the manifold V_{-4}. Consequently, if we follow the Kirby moves backwards from Figure 60 to Figure 52 (skipping Figures 55 and 56, as these are for the case when n is even), then we can conclude that $B_n \hookrightarrow V_{-4}$ for n odd. $\square$

Corollary 4.4. *For odd $n \geq 3$, the rational blow-up of $B_n \hookrightarrow V_{-4}$ is diffeomorphic to $V_{-4}\#(n-1)\overline{\mathbb{C}P^2}$.*

Similarly to the proof of Theorem 4.1 proving Corollary 4.3, the proof of Theorem 4.2 also proves Corollary 4.4. From the proof of Theorem 4.2, we can represent V_{-4} with the Kirby diagram in Figure 52, where we can see the $B_n \hookrightarrow V_{-4}$. If we rationally blow up this B_n, then we obtain the Kirby diagram in Figure 51, which by a sequence of Kirby moves gets us back to the diagram in Figure 44, which is precisely $V_{-4}\#(n-1)\overline{\mathbb{C}P^2}$.

4.2. **"Simple" embeddings.** It is worthwhile to remark that the embeddings of the B_ns in Theorems 4.1 and 4.2 are inherently different from the embeddings of $B_n \hookrightarrow E(m)_n$, as discussed in the beginning of section 4. As seen from Corollaries 4.3 and 4.4, the embeddings of $B_n \hookrightarrow V_{-n-1}, V_{-4}$ are such that if one rationally blows up those B_ns and then performs the regular blow-down $(n-1)$ times, then one gets back the manifolds V_{-n-1}, V_{-4} respectively. One could also do these two steps in reverse: if one starts with V_{-n-1}, V_{-4}, blows them up $(n-1)$ times and then rationally blows down the obtained C_n configuration, then one again obtains the manifolds V_{-n-1}, V_{-4} respectively. This is summarized in the following diagrams for the embeddings of $B_n \hookrightarrow V_{-n-1}$ for $n \geq 2$ and for $B_n \hookrightarrow V_{-4}$ for odd $n \geq 3$ respectively:

$$\begin{tikzcd} V_{-n-1} \arrow[r, "\text{RBU the } B_n"] \arrow[d, "\text{BU } (n-1) \text{ times}"] & V_{-n-1}\#(n-1)\overline{\mathbb{C}P^2} \arrow[d, "\text{BD } (n-1) \text{ times}"] \\ V_{-n-1}\#(n-1)\overline{\mathbb{C}P^2} \arrow[r, "\text{RBD the } C_n"] & V_{-n-1} \end{tikzcd}$$

$$\begin{tikzcd}
V_{-4} \arrow[r, "\text{RBU the } B_n"] \arrow[d, "\text{BU } (n-1) \text{ times}"] & V_{-4}\#(n-1)\overline{\mathbb{C}P^2} \arrow[d, "\text{BD } (n-1) \text{ times}"] \\
V_{-4}\#(n-1)\overline{\mathbb{C}P^2} \arrow[r, "\text{RBD the } C_n"] & V_{-4}
\end{tikzcd}$$

This is not the case with the embeddings of $B_n \hookrightarrow E(m)_n$, since the rational blow-ups of those B_n's result in $E(m)\#(n-1)\overline{\mathbb{C}P^2}$ (and not $E(m)_n\#(n-1)\overline{\mathbb{C}P^2}$) and so blowing down $(n-1)$ times yields the manifold $E(m)$ and not $E(m)_n$, the manifold we started with.

As a result, one can call an embedding of $B_n \hookrightarrow X$ "simple" if rationally blowing up and then blowing down $(n-1)$ times yields back the same 4-manifold X (the top and right arrows of the diagram below). Equivalently, an embedding $B_n \hookrightarrow X$ is "simple" if blowing up $(n-1)$ times followed by rationally blowing down the C_n, yields back the same 4-manifold X (the left and bottom arrows of the diagram below).

$$\begin{tikzcd}
X \arrow[r, "\text{RBU the } B_n"] \arrow[d, "\text{BU } (n-1) \text{ times}"] & X\#(n-1)\overline{\mathbb{C}P^2} \arrow[d, "\text{BD } (n-1) \text{ times}"] \\
X\#(n-1)\overline{\mathbb{C}P^2} \arrow[r, "\text{RBD the } C_n"] & X
\end{tikzcd}$$

It follows that the embeddings of of $B_n \hookrightarrow V_{-n-1}$ for $n \geq 2$ and for $B_n \hookrightarrow V_{-4}$ for odd $n \geq 3$ are "simple", whereas the embedding $B_n \hookrightarrow E(m)_n$ is not "simple". Therefore, one can ask the following question: Are there obstructions to embedding the B_ns in a "non-simple" way?

Nevertheless, Theorem 4.2 prevents one from finding an upper bound on n for a smooth 4-manifold X to contain an embedded B_n. In other words, one could not hope for something similar to Conjecture 1.2 to hold for smooth 4-manifolds. However, one can ask whether such a bound exists for "non-simple" embeddings of $B_n \hookrightarrow X$. For symplectic embeddings of B_n, the situation is very different, as will be shown in the section 5.

The Kirby diagrams in the proofs of Theorems 4.1 and 4.2 strongly suggest that the key to determining whether an embedding of a rational homology ball B_n is "simple" lies in analyzing how the extra sphere with self-intersection (-1) intersects with the spheres of the C_n configuration after one rationally blows up the B_n. For example, if one starts with $B_n \hookrightarrow V_{-n-1}$ for $n \geq 2$, and rationally blows it up, one obtains the Kirby diagram seen in Figure 30. In this case, the extra sphere with self-intersection (-1) intersects with the last sphere of self-intersection (-2) (S_{n-1} in Figure 25) in the C_n configuration. Likewise, if one starts with $B_n \hookrightarrow V_{-4}$ for odd $n \geq 3$, and rationally blows it up, one obtains the Kirby diagram seen in Figure 44. In this case, the extra sphere with self-intersection (-1) intersects with the first sphere of self-intersection (-2) (S_2 in Figure 25) in the C_n configuration. In the "non-simple" embedding case of $B_n \hookrightarrow E(m)_n$, if one rationally blows up those rational homology balls, then the extra sphere of self-intersection (-1) intersects with the first and last spheres of the C_n configuration (S_1 and S_{n-1}, respectively, in Figure 25), as seen in Figure 24. The intersection of this extra sphere of self-intersection (-1) with the spheres of the C_n configuration will play a central role in the study of symplectic embeddings of the rational homology balls B_n.

5. Symplectic embeddings of rational homology balls

We now turn to the question of when can one symplectically embed a rational homology ball B_n in a symplectic 4-manifold. As a warm-up, we give the following proposition:

Proposition 5.1. *There does not exist a symplectic embedding of $B_n \hookrightarrow E(2)$ for any $n \geq 2$.*

Proof. If n is even this is trivial since the B_ns aren't even spin manifolds. For n odd, we make the following observation about canonical classes: If there was a symplectic embedding of $B_n \hookrightarrow E(2)$, then the restriction of the canonical class of $E(2)$,

$K_{E(2)}$, to B_n must be the canonical class of B_n, K_{B_n}. In other words, we must have $K_{E(2)}|_{B_n} = K_{B_n}$. Likewise, the restriction of the canonical class of $E(2)$ to the boundary $\partial B_n = L(n^2, n-1)$ must correspond to the spin^c structure associated with the contact structure ξ_{std}. However, on one hand, since $E(2)$ is a $K3$ surface, its canonical class in trivial, $K_{E(2)} = 0$. On the other hand, in section 3.3 we computed that the contact structure ξ_{std} on $\partial B_n = L(n^2, n-1)$ corresponded to an element of order n in $H^2(L(n^2, n-1); \mathbb{Z}) \cong \mathbb{Z}/n^2\mathbb{Z}$. In addition, the homomorphism induced by the inclusion map $\iota : L(n^2, n-1) \to B_n$, on cohomology is ([Pa1]):

$$
\begin{aligned}
H^2(B_n; \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z} \quad &\to \quad H^2(L(n^2, n-1); \mathbb{Z}) \cong \mathbb{Z}/n^2\mathbb{Z} \\
x \quad &\to \quad n \cdot x\,.
\end{aligned} \tag{5.1}
$$

Therefore, the canonical class of B_n, K_{B_n}, is non-trivial as well, and is in fact a generator of $H^2(B_n; \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. As a result, $K_{E(2)_{|B_n}} \neq K_{B_n}$, preventing a symplectic embedding of $B_n \hookrightarrow E(2)$. $\square$

5.1. **Main theorem on symplectic embeddings of B_n.** In this section we present the statement of the main theorem on symplectic embeddings of B_n. Unlike the smooth embedding theorems, this theorem presents a negative result. Before we do so, we state the following crucial proposition and some definitions and terminology.

Proposition 5.2. *Let (X, ω) be a symplectic 4-manifold, such that:*

- $b_2^+(X) > 1$,
- $[c_1(X, \omega)] = -[\omega]$ *as cohomology classes and*
- $n \geq c_1^2(X, \omega) + 2$.

If there exists a symplectic embedding $B_n \hookrightarrow (X, \omega)$ and (X', ω') is the symplectic rational blow-up of (X, ω), then there exists an embedded symplectic sphere $\Sigma_{-1} \subset (X', \omega')$, and a linear plumbing configuration $C_n \subset (X', \omega')$ of symplectic spheres S_j, $1 \leq j \leq n-1$, such that:

- $[\Sigma_{-1}]^2 = -1$,
- $[S_1]^2 = -n-2$ *and* $[S_j]^2 = -2$ *for* $2 \leq j \leq n-1$ *(see Figure 25) and*
- Σ_{-1} *intersects the spheres* S_j, $1 \leq j \leq n-1$ *positively and transversally.*

Definition 5.3. We call a symplectic embedding of $B_n \hookrightarrow (X, \omega)$ to be of *type* $\langle \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1} \rangle$, where $\alpha_j \in \mathbb{Z}_{\geq 0}$, if there exists an embedded symplectic sphere, $\Sigma \subset X'$, with $[\Sigma]^2 = -1$, such that it intersects positively and transversally with the spheres S_j, $1 \leq j \leq n-1$, of the C_n configuration in X' and α_j is the number of those positive transverse intersections.

Definition 5.4. Let $\mathcal{A}$ be the set of $(n-1)$-tuples $\langle \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1} \rangle$ such that:

(1) $\alpha_j \neq 0$ for at least one j, where $2 \leq j \leq n-1$, or

(2) $\alpha_1 \geq n$, or

(3) $\alpha_1 = 1$ and $\alpha_j = 0$ for $2 \leq j \leq n-1$.

We will call a symplectic embedding $B_n \hookrightarrow X$ to be of *type* $\mathcal{A}$ if it is of type $\langle \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1} \rangle \subset \mathcal{A}$.

Definition 5.5. Let $\mathcal{E}_k$ denote the $(n-1)$-tuple $\langle k, 0, 0, \ldots, 0 \rangle$ for $2 \leq k \leq n-1$.

We note that Proposition 5.2 implies that a symplectic embedding $B_n \hookrightarrow (X, \omega)$ (for $b_2^+(X) > 1$, $[c_1(X, \omega)] = -[\omega]$ and $n \geq c_1^2(X, \omega) + 2$) will always be of type $\langle \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1} \rangle$, for some $(n-1)$-tuple $\langle \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1} \rangle$ with $\alpha_j \in \mathbb{Z}_{\geq 0}$. Moreover, any $(n-1)$-tuple $\langle \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1} \rangle$ with $\alpha_j \in \mathbb{Z}_{\geq 0}$ will be in at least one of the sets $\mathcal{A}, \mathcal{E}_k$, $2 \leq k \leq n-1$.

Theorem 5.6. *If* $B_n \hookrightarrow (X, \omega)$ *is a symplectic embedding, where* (X, ω) *is a symplectic* 4*-manifold, such that:*

- $b_2^+(X) > 1$,
- $[c_1(X, \omega)] = -[\omega]$ *as cohomology classes,*
- $n \geq c_1^2(X, \omega) + 2$ *and*

- *$\mathcal{B}as_X = \{\pm c_1(X,\omega)\}$, ($\mathcal{B}as_X$ denotes the set of Seiberg-Witten basic classes of X,)*

then it cannot be of type $\mathcal{A}$ or of type $\mathcal{E}_k$, $k \geq c_1^2(X,\omega)+2$.

Remark 5.7. The condition $[c_1(X,\omega)] = -[\omega]$ holds for surfaces of general type X, with the canonical class K_X ample. The ampleness implies that for all curves C in X, we have that $c_1(X,\omega)\cdot[C] < 0$, implying that there are no (-1) or (-2) curves in X. For a symplectic 4-manifold X, the condition $[c_1(X,\omega)] = -[\omega]$ implies that there are no symplectic spheres S with self-intersection (-1) or (-2): since for a symplectic sphere S, we have $\int_S \omega > 0$ which implies $c_1(X)\cdot[S] < 0 \Rightarrow [S]^2 < -2$ by the adjunction inequality. Additionally, the condition of (X,ω) having only one Seiberg-Witten basic class (up to sign), is also true of all surfaces of general type. Consequently, these symplectic 4-manifolds are meant to mimic surfaces of general type as much as they can.

We will prove this theorem in four steps. In ***Step 1***, section 5.3.1, we will prove Proposition 5.2. In ***Step 2***, section 5.3.2, using the existence of the sphere Σ_{-1} from Proposition 5.2, we construct a specific homology cycle γ, and compute $c_1(X,\omega)\cdot\gamma$ in terms of the intersection pattern of Σ_{-1} with the spheres of the C_n configuration. In ***Step 3***, section 5.3.3, we show that if $c_1(X,\omega)\cdot\gamma > 0$, then $\omega\cdot\gamma > 0$, thus contradicting the $[c_1(X,\omega)] = -[\omega]$ assumption on (X,ω). As a result, we will show show that $B_n \hookrightarrow (X,\omega)$ cannot be of type $\mathcal{A}_1 \subset \mathcal{A}$, where $\mathcal{A}_1$ is the set of $(n-1)$-tuples corresponding to $c_1(X,\omega)\cdot\gamma > 0$. In ***Step 4***, section 5.3.4, we show that if $c_1(X,\omega)\cdot\gamma \leq 0$, then this violates certain adjunction inequalities or forces X to have additional Seiberg-Witten basic classes, thus preventing $B_n \hookrightarrow (X,\omega)$ to be of type $(\mathcal{A}-\mathcal{A}_1)$ and $\mathcal{E}_k$, $k \geq c_1^2(X,\omega)+2$.

Remark 5.8. In section 5.3.5, we give explicit examples of symplectic embeddings of $B_n \hookrightarrow (X,\omega)$ of type $\mathcal{E}_2$ for n odd. In these examples, we always have $n <$

$3+\frac{4}{3}c_1^2(X,\omega)$. However it is not entirely clear how to construct manifolds (X,ω), such that they will have embeddings $B_n \hookrightarrow (X,\omega)$ of type $\mathcal{E}_k$, with $3 \leq k \leq c_1^2(X,\omega)+1$. We do propose the following conjecture:

Conjecture 5.9. *For n odd, there exist symplectic embeddings of $B_n \hookrightarrow (X,\omega)$ of type $\mathcal{E}_k$, with $3 \leq k \leq c_1^2(X,\omega)+1$, for symplectic 4-manifolds (X,ω) with $b_2^+(X) > 1$, $n \geq c_1^2(X,\omega)+2$ and $\mathcal{B}as_X = \{\pm c_1(X,\omega)\}$. Moreover, for such embeddings, we will have a bound of the type $n < A(\chi_h(X)) + B(c_1^2(X))$, for some A and B.*

(Note, χ_h denotes the holomorphic Euler number which equals to $(b_2^+ + 1)/2$ for manifolds with $b_1 = 0$.)

Before we present a proof of Theorem 5.6, we will present some additional background material below in section 5.2.

5.2. More background material.

5.2.1. *Toric and almost-toric fibrations of symplectic 4-manifolds.* In [Sy1], Symington showed that the rational blow-down construction can be performed in the symplectic category. She did this by describing the symplectic structure of C_n and a collar neighborhood of ∂B_n with the help of toric fibrations. In [Sy2], she generalized this construction to show that the generalized rational blow-down (see section 6) can also be performed in the symplectic category. In [Sy3], she presented a way of describing symplectic 4-manifolds through almost-toric fibrations and used this to prove the existence of the symplectic rational blow-down in a less cumbersome manner than using just toric fibrations. In what follows, we present a brief review of toric and almost-toric fibrations of symplectic 4-manifolds, including some key results and examples, which will be used in *Step 3* (Section 5.3.3) in the proof of Theorem 5.6.

The goal of toric and almost-toric fibrations of symplectic 4-manifolds is to be able to depict various topological and symplectic properties of these manifolds with polytopes and curves in plane. The basis for doing this comes from a theorem of Delzant:

Theorem 5.10. [De] *If a closed symplectic manifold* (M^{2n}, ω) *is equipped with an effective Hamiltonian* n*-torus action, then the image of the moment map* Δ *determines the manifold* M*, its symplectic structure* ω *and the torus action.*

Additionally, we have the following key result on Hamiltonian torus actions:

Theorem 5.11. [At, GS] *The moment map image* Δ *for a Hamiltonian* k*-torus action on a closed symplectic manifold* (M, ω) *is a convex polytope.*

When $k = n$, the manifold (M^{2n}, ω) is called toric. For our purposes we will only be dealing with the case $n = 2$, and while several of the following results hold in any even dimension, we will only state them for $n = 2$. The main goal of [Sy3], with the almost-toric fibrations is to extend the above two theorems to work for a larger class of symplectic 4-manifolds, and generalize the class of moment-map images.

Since the symplectic form vanishes on the fibers of a moment map, implying that the regular fibers are Lagrangian submanifolds, the moment map actually provides us with a Lagrangian fibration:

Definition 5.12. [Sy3] A projection $\pi : (M^4, \omega) \to B^2$ is a *Lagrangian fibration* if it restricts to a regular Lagrangian fibration (locally trivial fibration where the fibers are Lagrangian) over an open dense set $B_0 \subset B$.

The most basic example is $\pi : (\mathbb{R}^2 \times T^2, \omega_0) \to \mathbb{R}^2$, with ω_0 the standard symplectic structure, which serves as a model for all other examples. The goal is to make use of the standard lattice Λ_0 on the tangent bundle $T\mathbb{R}^2$, spanned by $\left\{\frac{\partial}{\partial p_i}\right\}$ and $\left\{\frac{\partial}{\partial q_i}\right\}$, where (p, q) are the standard coordinates on $\mathbb{R}^2 \times T^2$. In relation to this, Symington shows the following:

Theorem 5.13. [Sy3] *If $\pi : (M, \omega) \to B$ is a regular Lagrangian fibration then there are lattices $\Lambda \subset TB$, $\Lambda^* \subset T^*B$ and Λ^{vert} in the vertical bundle of TM (induced by π) that, with respect to standard local coordinates, are the standard lattice, its dual, and the standard vertical lattice.*

This induced lattice Λ on the tangent bundle of the base B, as above, gives B an *integral affine structure* $\mathcal{A}$.

Proposition 5.14. [Sy3] *An n-manifold B admits an integral affine structure if and only if it can be covered by coordinate charts $\{U_i, h_i\}$, $h_i : U_i \to \mathbb{R}^n$ such that the map $h_j \circ h_i^{-1}$, wherever defined, is an element of $AGL(n, \mathbb{Z})$, i.e. a map of the form $\Phi(x) = Ax + b$ where $A \in GL(n, \mathbb{Z})$ and $b \in \mathbb{R}^n$.*

Symington denotes the toric (and almost-toric) bases with $(B, \mathcal{A}, \mathcal{S})$, where B is the polytope base in $\mathbb{R}^n$ (see Theorem 5.11), $\mathcal{A}$ is an *integral affine structure*, and $\mathcal{S}$ is a natural stratification of the base B: the l-stratum is the set of points $b \in B$ such that $\pi^{-1}(b)$ is a torus of dimension l. Additionally, $\partial_R B$ denotes the collection of all the k-strata, with $k < n$, which is the *reduced boundary* of the base $(B, \mathcal{A}, \mathcal{S})$. Symington gives the following definition of the *toric fibration and base*:

Definition 5.15. [Sy3] A Lagrangian fibration $\pi : (M^4, \omega) \to (B, \mathcal{A}, \mathcal{S})$ is a *toric fibration* if there is a Hamiltonian 2-torus action and an immersion $\Phi : (B, \mathcal{A}) \to (\mathbb{R}^2, \mathcal{A}_0)$ such that $\Phi \circ \pi$ is the corresponding moment map and $\mathcal{S}$ is the induced stratification. In this case we call $(B, \mathcal{A}, \mathcal{S})$ a *toric base.*

Since we are looking to represent symplectic 4-manifolds, we will be working with bases of dimension 2, and with 2, 1 and 0-strata. In other words, the 1-stratum are the edges of our polytope B in the plane, and the 0-stratum are its vertices. Consequently, Symington's goal was to put the appropriate conditions on the base $(B, \mathcal{A}, \mathcal{S})$ to ensure that it determines a unique symplectic 4-manifold. To reconstruct a symplectic

4-manifold from a toric base $(B, \mathcal{A}, \mathcal{S})$, one can start with a regular Lagrangian fibration over $(B, \mathcal{A})$ and collapse certain fibers to get the desired stratification $\mathcal{S}$. Symington does this with the help of *symplectic boundary reduction*, introduced in [Sy1], which is defined in the proposition below:

Proposition 5.16. *Let (M, ω) be a symplectic manifold with boundary such that a smooth component Y of ∂M is a circle bundle over a manifold Σ. Suppose also that the tangent vectors to the circle fibers lie in the kernel of $\omega|_Y$. Then there is a projection $\rho : (M, \omega) \to (M', \omega')$ and an embedding $\phi : \Sigma \to M'$ such that $\rho(Y) = \phi(\Sigma)$, $\rho|_{M-Y}$ is a symplectomorphism onto $M' - \phi(\Sigma)$ and $\phi(\Sigma)$ is a symplectic submanifold. The manifold $(M', \omega') = \rho(M, \omega)$ is the* ***symplectic boundary reduction*** *of (M, ω) along Y.*

Connecting the above proposition to the toric bases $(B, \mathcal{A}, \mathcal{S})$, Symington gives the following definition:

Definition 5.17. Given a toric fibration $\pi : (M^4, \omega) \to (B, \mathcal{A}, \mathcal{S})$, the *boundary recovery* is the unique Lagrangian fibered manifold $(B \times T^2, \omega_0)$ that yields (M, ω) via boundary reduction.

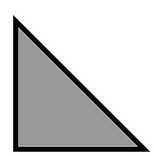


FIGURE 61. Toric model of $\mathbb{C}P^2$

Example 5.18. A basic example is the toric base for a symplectic 4-manifold diffeomorphic to $\mathbb{C}P^2$, which is a simple triangle with vertices on $(0, 0)$, $(1, 0)$ and $(0, 1)$, as depicted in Figure 61, with the bold edges representing the 1-stratum. This base represents $\mathbb{C}P^2$ as the boundary reduction of (B^4, ω_0), where the circles of the Hopf fibration are collapsed.

In reading such diagrams, it is important to remember that the pre-image of each interior point in the diagram, is a torus, $S^1 \times S^1$, the pre-image of each point on the thick edges of the diagram (the 1-stratum) is a circle S^1, and the pre-image of each vertex in the diagram is just a point. Before giving another example, that will be directly relevant to rational blow-downs, we first introduce the following important element of toric bases, having to do with the interpretation of the slopes of the lines in these diagrams:

Definition 5.19. [Sy3] Let $\pi : (M, \omega) \to (B, \mathcal{A}, S)$ be a toric fibration and γ a compact embedded curve with one endpoint b_1 in the 1-stratum of $\partial_R B$ (both the 1 and 0-stratum in this case) and such that $\gamma - \{b_1\} \subset B_0 = B - \partial_R B$. Let b_0 be the other endpoint of γ. The *collapsing class*, with respect to γ, for the smooth component of $\partial_R B$ containing b_1 is the primitive class $\mathbf{a} \in H_1(F_{b_0}; \mathbb{Z})$ that spans the kernel of $\iota_* : H_1(F_{b_0}; \mathbb{Z}) \to H_1(\pi^{-1}(\gamma); \mathbb{Z})$, where ι is the inclusion map. Corresponding to the *collapsing class* is the *collapsing covector*, with respect to γ, which is the primitive covector $v^* \in T^*_{b_0} B$ that determines vectors $v(x) \in T^{vert}_x M$ for each $x \in \pi^{-1} b_0$ such that the integral curves of this vector field represent $\mathbf{a}$.

Example 5.20. Another example is the toric base for a symplectic neighborhood of the spheres in the C_3 configuration, as seen in Figure 62. (Note, the actual slopes of the lines in Figure 62 have been exaggerated for visual effect.) The horizontal edge on the bottom of the diagram, E_1, represents the symplectic sphere S_1 in the C_3 configuration, with $[S_1]^2 = -5$. The slanted edge on the bottom, E_2, represents the symplectic sphere S_2 in the C_3 configuration, with $[S_2]^2 = -2$.

The lengths of these horizontal edges correspond to the areas of these symplectic spheres (see [Sy1, Sy2]). The self-intersection numbers of the spheres S_1 and S_2 can

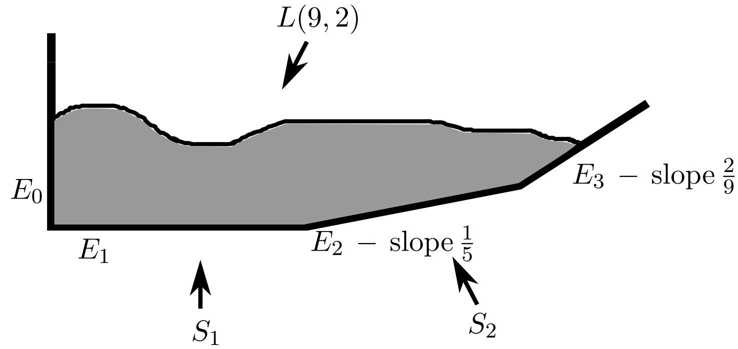


FIGURE 62. Toric model for C_3

be read off from the slopes of the lines in the diagram. The slopes of edges E_2 and E_3 in Figure 62 are $\frac{1}{5}$ and $\frac{2}{9}$, respectively. The collapsing covector for a path in the base hitting E_0 is $v_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the collapsing covector for a path hitting E_2 is $v_2 = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$. It can be shown, that the self-intersection number of the sphere S_1, represented by the edge E_1, can be computed by taking the cross product of the collapsing covectors v_0 and v_2: $v_2 \times v_0 = -5$, giving us $[S_1]^2 = -5$ (see [Sy1, Sy2]). Similarly, we can compute the self-intersection number of the sphere S_2, represented by the edge E_2, by taking the cross product of the collapsing covectors v_1 and v_3, corresponding to paths hitting the edges E_1 and E_3. We have $v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $v_3 = \begin{bmatrix} -2 \\ 9 \end{bmatrix}$, giving us $v_3 \times v_1 = -2$ which implies $[S_2]^2 = -2$.

The (thin) curve on top is the boundary $\partial C_3 = L(9,2)$. Note, that this curve must be convex, to insure that we get symplectic convexity on C_3, $\partial C_3 = L(9,2)$. The fibers above this curve do not collapse, as they are not in the 1-stratum, thus the pre-image of every point on this curve is a torus $S^1 \times S^1$. When this curve hits the edge E_0, the collapsing covector is $v_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. When this curve hits the edge E_3, the collapsing covector is $v_3 = \begin{bmatrix} -2 \\ 9 \end{bmatrix}$. As a result, the collection of all the pre-images of the points on this curve forms the lens space $L(9,2)$.

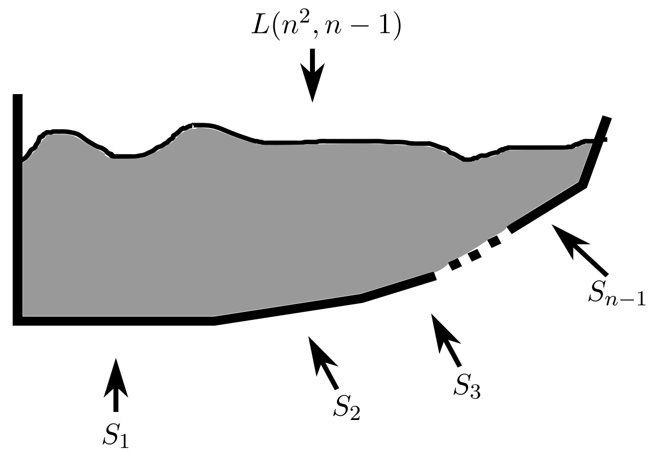


FIGURE 63. Toric model for C_n

Example 5.21. Similarly to Example 5.20, we can construct a toric fibration of the C_n configuration of spheres, (see Figure 63). In this diagram, the slopes of the edges are $0, \frac{1}{n+2}, \frac{2}{2n+3}, \frac{3}{3n+4}, \ldots, \frac{n-1}{n^2}$, thus the corresponding collapsing covectors are: $v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ n+2 \end{bmatrix}, v_3 = \begin{bmatrix} -2 \\ 2n+3 \end{bmatrix}, v_4 = \begin{bmatrix} -3 \\ 3n+4 \end{bmatrix}, \ldots, v_n = \begin{bmatrix} 1-n \\ n^2 \end{bmatrix}$. Consequently, we have $v_{i+1} \times v_{i-1} = [S_i]^2$, giving us the desired self-intersection numbers of the spheres S_i.

As in the previous example, the pre-image of the (thin) curve on the top of the diagram is the boundary $\partial C_n = L(n^2, n-1)$, since the collapsing covectors on both endpoints of the curve are $v_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_n = \begin{bmatrix} 1-n \\ n^2 \end{bmatrix}$.

Symington proves (Theorem 3.19, [Sy3]) that such diagrams of toric bases $(B, \mathcal{A}, \mathcal{S})$, determine unique toric manifolds, presented as the boundary reduction of $(B \times T^2, \omega_0)$

(assuming certain technical conditions, see [Sy3] for details, all of which are satisfied for the examples we present). The uniqueness of the toric manifold fibering over the base $(B, \mathcal{A}, \mathcal{S})$, was shown earlier by [BM].

One can push these diagrams further, to depict Lagrangian fibrations with (nodal) singularities:

Definition 5.22. [Sy3] A nondegenerate Lagrangian fibration $\pi : (M, \omega) \to B$ of a symplectic 4-manifold is an *almost-toric fibration* if it is a nondegenerate topologically stable fibration with no hyperbolic singularities (e.g. a fibration with a nodal singularity). A triple $(B, \mathcal{A}, \mathcal{S})$ is an *almost-toric base* if it is the base of such a fibration. A symplectic 4-manifold equipped with such a fibration is an *almost-toric manifold*.

Thus if $\{s_i\} \subset B$ are the images of such singularities, then $\mathcal{A}$ is the affine structure on $B - \{s_i\}$. Generally, a Lagrangian fibration can be arranged such that nodal singularities occur in distinct fibers. Also, a nodal fiber is the singular fiber of a Lefschetz fibration, and its neighborhood is diffeomorphic to $T^2 \times D^2$ with a (-1)-framed two-handle attached along a simple closed curve in $T^2 \times \{x\}$. One can also compute the topological monodromy around the nodal fiber with respect to the basis $\{[\gamma_1], [\gamma_2]\} \in H_1(F_b; \mathbb{Z})$:

$$\Psi(\gamma) \;=\; A_{(1,0)} \;=\; \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

If we choose a different basis for $H_1(F_b; \mathbb{Z})$, then we conjugate the matrix $A_{(1,0)}$, giving us the following monodromy matrix with eigenvector (a, c):

$$A_{(a,c)} \;=\; \begin{pmatrix} 1 - ac & a^2 \\ -c^2 & 1 + ac \end{pmatrix}.$$

This leads us to the following definition and lemma:

Definition 5.23. [Sy3] Let $\pi : (M, \omega) \to B$ be an almost-toric fibration with a node at s. Let η be an embedded curve with endpoints at s and a point $b \in B_0 = B - \partial_R B$ such that $\eta - \{s\} \subset B_0$ contains no other nodes. A *vanishing class* in $H_1(F_b; \mathbb{Z})$, associated to s and η, is the class whose representatives bound a disk in $\pi^{-1}(\eta)$. The *vanishing covector* $w^* \in T_b^* B$ is the primitive covector that determines vectors $w(x) \in T_x^{vert} M$ for each $x \in \pi^{-1} b$ such that the integral curves of this vector field represent the vanishing class.

Lemma 5.24. [Sy3] *Suppose γ is a positively oriented loop based at b that is the boundary of a closed neighborhood of s containing η. Then the vanishing class is the unique class (up to scale) that is preserved by the monodromy along γ. With respect to the basis for $H_1(F_b; \mathbb{Z})$ for which the monodromy matrix is $A_{(a,c)}$, the vanishing class is the class (a, c).*

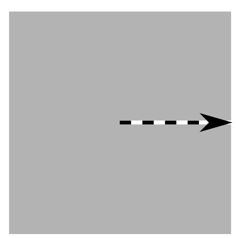


FIGURE 64. Almost toric base

Notice, that given such an almost-toric fibration $\pi : (M, \omega) \to B$ with a node at s, the fibration over $B - \{s\}$ is regular and has an induced affine structure $\mathcal{A}$. However, there is non-trivial monodromy around the node s, thus there is no affine immersion of $(B - \{s\}, \mathcal{A})$ into $(\mathbb{R}^2, \mathcal{A}_0)$. To salvage this, we can remove a ray R, based at the node s, from the base B, giving us an immersion of $(B - R, \mathcal{A})$ into $(\mathbb{R}^2, \mathcal{A}_0)$. An example of such a base B with a removed ray R is seen in Figure 64. If one chooses a different ray R' then, the the immersion of $(B - R', \mathcal{A})$ into $(\mathbb{R}^2, \mathcal{A}_0)$, will have an image that is a neighborhood of an origin without a sector of internal angle between 0 and π. The angle will be 0 (as seen in Figure 64) if the ray R lies within an *eigenline*:

Definition 5.25. The *eigenline* L through a node s is the unique maximal affine linear immersed one-manifold through the node for which there is a regular point $b \in L$, arbitrarily close to s, such that the affine monodromy along an arbitrary small loop around s and based at b preserves $T_bL \subset T_bB$. An eigenray is either of the two maximal affine linear submanifolds that has an endpoint at the node and is a subset of the *eigenline*.

Symington shows, that any integral affine punctured plane $(V, \mathcal{A})$ will be isomorphic to the one depicted in Figure 64, where the ray R is the eigenray $(1, 0)$. Consequently, we can model an almost-toric manifold with bases B containing nodes s_i:

Definition 5.26. [Sy3] An *integral affine manifold with nodes* $(B, \mathcal{A})$ is a two-manifold B equipped with an integral affine structure on $B - \{s_i\}$ such that each s_i has a neighborhood U_i such that $(U_i - s_i, \mathcal{A})$ is affine isomorphic to a neighborhood of the puncture in $(V^k, \mathcal{A}^k)$ (if the node has multiplicity k).

Theorem 5.27. [Sy3] *Consider a triple $(B, \mathcal{A}, \mathcal{S})$ such that $(B, \mathcal{A})$ is an integral affine manifold with nodes $\{s_i\}_{i=1}^N$. Then $(B, \mathcal{A}, \mathcal{S})$ is an almost-toric base if and only if every point in $B - \{s_i\}_{i=1}^N$ has a neighborhood that is a toric base.*

Symington also defined various operations, like the *nodal slide* and the *nodal trade* to get from one almost toric base $(B, \mathcal{A}, \mathcal{S})$ to another $(B', \mathcal{A}', \mathcal{S}')$, with both representing the same manifold with isotopic symplectic structures. Now we are ready to describe the almost-toric base for the rational homology balls B_n:

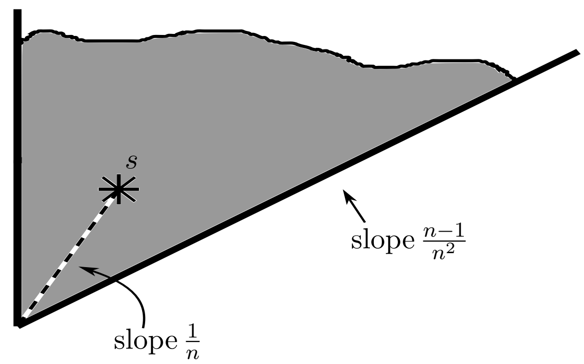


FIGURE 65. Almost toric base for B_n

Example 5.28. Figure 65 depicts an almost-toric base for the rational homology balls B_n, in this diagram, the ray R has a of slope of $\frac{1}{n}$, corresponding to the eigenvector $(n, 1)$ of the monodromy

$$A_{(n,1)} = \begin{pmatrix} 1-n & n^2 \\ -1 & 1+n \end{pmatrix}.$$

thus making $\begin{bmatrix} -1 \\ n \end{bmatrix}$ be the vanishing covector of the node s. The slope of the line on the right is $\frac{n-1}{n^2}$, therefore the preimage of the thin line on the top of the diagram is $L(n^2, n-1)$ as was the case for the toric diagram for C_n.

Symington then proves that the symplectic rational blow-down can be performed in the symplectic category, by simply removing the images of the neighborhoods of the symplectic spheres from the toric model of C_n (Figure 63) and gluing below it, the almost-toric model for B_n (Figure 65). They match up, since the slopes of the right-most edge is $\frac{n-1}{n^2}$, as illustrated in Figure 66.

It is useful for our purposes to illustrate where on this almost-toric model of B_n can we "see" the image of the "Lagrangian cores" $\mathcal{L}_{n,1}$ of the rational homology B_n, as introduced in the *symplectic rational blow-up* construction in section 3.4.1. Before we

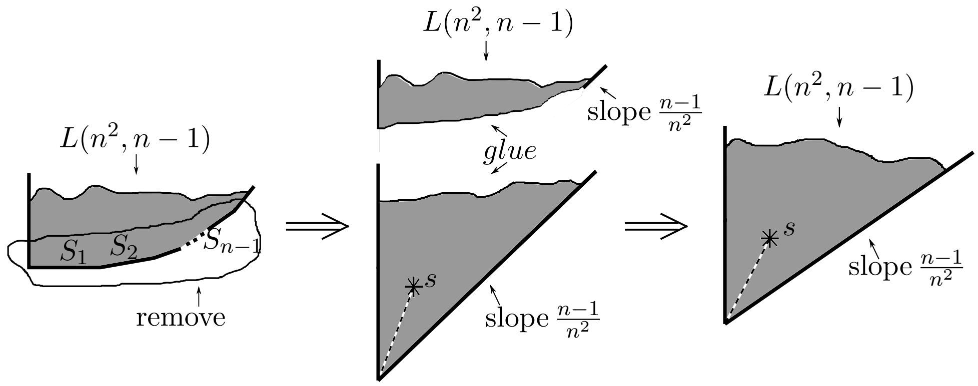


FIGURE 66. Rational blow-down in almost-toric diagrams

do this, we must first introduce the concept of *visible surfaces* in these almost-toric fibrations, as was done in [Sy3].

If one draws a curve ν in an (almost)-toric base B, then the pre-image of every point $b \subset B_0$ in the curve will be a torus $F_b \cong S^1 \times S^1$. However, if for every point b in the curve we choose a closed curve in $F_b \cong S^1 \times S^1$, then the entire collection of those closed curves over all points in the curve ν could potentially be a surface in the original 4-manifold. This is precisely what *visible surfaces* are, they are a coherent collection of such closed curves, in the pre-images of the points in a toric (almost-toric) base. Here is a more precise definition that Symington gives:

Definition 5.29. [Sy3] A *visible surface* Σ_ν in an almost-toric fibered manifold $\pi : (M, \omega) \to (B, \mathcal{A}, \mathcal{S})$ is an immersed surface whose image is an immersed (connected) curve ν with transverse self-intersections such that $\pi|_{\Sigma_\nu \cap \pi_{-1}(B_0)}$ is a submersion onto $\nu \cap B_0$, any non-empty intersection of Σ_ν with a regular fiber is a union of affine circles, and no component of $\partial\Sigma_\nu$ projects to a node.

The following are the conditions on curves in the base to represent a *visible surface* and for a curve ν to represent a unique surface Σ_ν:

Definition 5.30. [Sy3] Given an immersed curve $\nu : I \to (B, \mathcal{A}, S)$, let $\{\nu_i\}_{i=1}^k$ be the continuous (and connected) components of $\nu|_{\nu^{-1}(B - \partial_R B)}$. A primitive class $\mathbf{a}_i$ in

$H_1(\pi^{-1}(\nu_i);\mathbb{Z})$ (such that $\pi_*\mathbf{a}_i = 0$ if ν_i is a loop) is *compatible* with ν if all of the following are satisfied:

(1) $\mathbf{a}_i$ is the *vanishing class* of every node in ν,

(2) $|\mathbf{a}_i \cdot \mathbf{c}| \in \{0, 1\}$ for each $\mathbf{c}$ that is the collapsing class, with respect to $\overline{\nu_i}$, for a component of the 1-stratum of $\partial_R B$ that intersects $\overline{\nu_i}$,

(3) $|\mathbf{a}_i \cdot \mathbf{c}| = 1$ if $\overline{\nu_i}$ intersects the 1-stratum non-transversally,

(4) $|\mathbf{a}_i \cdot \mathbf{d}| = 1$ for each $\mathbf{d}$ that is one of the two collapsing classes at a vertex contained in the closure of ν_i. (Here, $\cdot$ is the intersection pairing in $H_1(\pi^{-1}(\nu_i), \mathbb{Z})$ and $\overline{\nu_i}$ is the closure of ν_i.)

Theorem 5.31. [Sy3] *Suppose $(B, \mathcal{A}, \mathcal{S})$ is an almost-toric base such that each node has multiplicity one. An immersed curve $\nu : I \to (B, \mathcal{A}, \mathcal{S})$ with transverse self-intersections and a set of compatible classes $\{\mathbf{a}_i\}_{i=1}^k$ together determine a visible surface Σ_ν such that for each $b \in \nu_i$,*

$$\iota_*[\Sigma_\nu \cap F_b] = \mathbf{a}_i \tag{5.2}$$

where $\iota : F_b \to \pi^{-1}(\nu_i)$ is the inclusion map. (Note, we will not define the "multiplicity" of a node here; all of the nodes that we will work with have "mutliplicity" one, for details see [Sy3]*.) The surface Σ_ν is unique up to isotopy among visible surfaces in the preimage of ν that satisfy equation 5.2. Furthermore, no such surface exists if the classes $\mathbf{a}_i$ are not compatible with ν.*

Note, that given such conditions for a visible surface, Σ_ν must be a sphere, disk, cylinder or torus. Therefore, if we want to "see" a Lagrangian core $\mathcal{L}_{n,1}$ in the almost-toric base for B_n, we can only really "see" where $\mathcal{L}_{n,1}$ is an embedding, in other words, a Lagrangian disk in $\mathcal{L}_{n,1}$, right before the edge of the disk hits the singular part of $\mathcal{L}_{n,1}$. First, we have to say a few more words on the symplectic areas of Σ_ν and the covectors representing a primitive class $\mathbf{a}_i \in H_1(\pi^{-1}(\nu_i);\mathbb{Z})$.

To each primitive class $\mathbf{a}_i \in H_1(\pi^{-1}(\nu_i);\mathbb{Z})$ there is corresponding *compatible vector* $v_i \in \mathbb{R}^2$ such that the integral curves of the vector field $v_i\frac{\partial}{\partial q} \subset \Lambda^{vert}$ represent $\mathbf{a}_i$. If v and w are compatible vectors for primitive classes $\mathbf{a}$ and $\mathbf{b}$ respectively, then $|\mathbf{a}\cdot\mathbf{b}| = |v \times w| = |det(vw)|$. Symington also shows that if curves ν_1 and ν_2 intersect transversally at a point $b \in B_0$ and Σ_{ν_1} and Σ_{ν_2} intersect transversally in F_b, then Σ_{ν_1} intersects Σ_{ν_2} in $|v_1 \times v_2|$ points where the signs of all intersections is $det(u_1u_2)det(v_1v_2)$. Here, v_i are the compatible vectors of ν_i and the u_i are the tangent vectors of ν_i at the point b.

In [Sy3], it is proved that one can compute the symplectic area of the visible surfaces as follows:

Proposition 5.32. *Let $\nu : I \to (B, \mathcal{A}, \mathcal{S})$ be a parameterized immersed curve and $\{v_i\}_{i=1}^N$ a set of co-oriented compatible vectors in a base diagram that define an oriented surface Σ_ν. The the (signed) area of Σ_ν is:*

$$Area(\Sigma_\nu) = \int_{\Sigma_\nu} \omega = 2\pi \int_0^1 \nu'(t)\cdot v(t)dt \tag{5.3}$$

where $v(t) = v_i$, if $\nu(t) \in \nu_i$ and for other values of t (when $\nu \subset \partial_R B$) $v(t)$ is an integral vector such that $u(t) \times v(t) = 1$ for some integral vector $u(t) = \lambda\nu'(t)$, $\lambda > 0$.

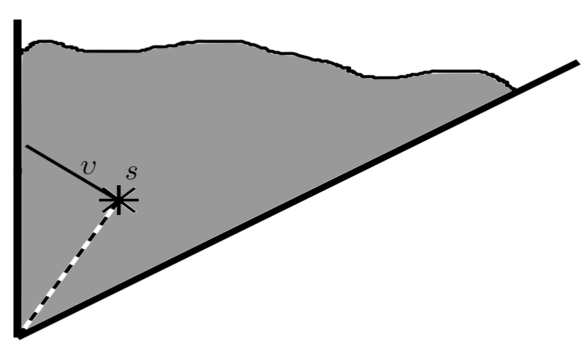


FIGURE 67. "visible" $\mathcal{L}_{n,1}$ in almost toric base for B_n

Remark 5.33. Proposition 5.32 implies that in order for a visible surface Σ_v to be Lagrangian, we must have that v is a straight line. The line v in Figure 67, extending from the node and (almost) hitting the left edge of the 1-stratum, represents a Lagrangian visible surface Σ_v. Since the line v hits a node, its compatible covector must correspond to the vanishing covector of the node, which is $v = \begin{bmatrix} -1 \\ n \end{bmatrix}$. Notice, if v were to actually hit the left edge of the 1-stratum, then this would violate condition (2) of Definition 5.30, since $|v \times c| = n$, where c is the collapsing covector of the left edge of the 1-stratum. As a result, Σ_v can represent a Lagrangian core $\mathcal{L}_{n,1}$ of B_n, as introduced in section 3.4.1, since the winding number of $\gamma : S^1 \hookrightarrow \mathcal{L}_{n,1}(\partial D)$ in Definition 3.7 is n.

5.2.2. *Review of Seiberg-Witten invariants and basic classes.* Here we give a brief overview of Seiberg-Witten invariants and basic classes, and state some relevant results. For a full description of Seiberg-Witten invariants see [Mo], and for a short overview see [GS], section 2.4 (which this summary is based on). The Seiberg-Witten invariant is a powerful invariant of smooth manifolds. More precisely, these are invariants of a smooth 4-manifold together with a *spin*^{c} structure (see Definition 5.34 below). We let X be a smooth, closed, oriented 4-manifold, with $b_2^+(X) > 1$ odd.

Definition 5.34. Let $U(2)$ be the group of 2×2 unitary matrices, then

$$Spin^c(4) = \{(A, B) \in U(2) \times U(2) \,|\, \det(A) = \det(B)\}\,. \tag{5.4}$$

A *spin*^{c} *structure* $\mathfrak{s}$ for X is given by fixing a principal $Spin^c(4)$-bundle, $P_{Spin^c(4)} \to X$, together with an identification $c : P_{Spin^c} \times_\rho SO(4) \cong P_{SO(4)}$, where $\rho : Spin^c \to SO(4)$ on the fibers. (Note: $SO(4) \cong SU(2) \times SU(2)/\{\pm(I, I)\}$.)

Given a *spin*^{c} structure $\mathfrak{s}$, we can associate to it a *determinant line bundle* L: to $(A, B) \in Spin^c$ we can associate $\det(A)$, giving us a homomorphism $\alpha : Spin^c \to S^1$, from which we can construct a line bundle $L = P_{Spin^c(4)} \times_\alpha \mathbb{C}$.

Definition 5.35. The set of *characteristic elements* of X (as above) is:

$$\mathcal{C}_X = \left\{ K \in H^2(X;\mathbb{Z}) | K \equiv w_2(X) (\text{mod} 2) \right\} . \tag{5.5}$$

Proposition 5.36. [GS] *Suppose that $\mathfrak{s}$ is a given spin^c structure with determinant line bundle L. Then the first Chern class $c_1(L) \in H^2(X;\mathbb{Z})$ of L satisfies $c_1(L) \equiv w_2(X)\,(mod 2)$, hence it is a characteristic element. For every characteristic element $K \in \mathcal{C}_X$, there is a spin^c structure with determinant line bundle L satisfying $c_1(L) = K$. Every oriented (possibly noncompact) 4-manifold admits a spin^c structure. If X is simply connected (or more generally $H^2(X;\mathbb{Z})$ has no 2-torsion) the determinant line bundle determines the spin^c structure, and the set of spin^c structures $\mathcal{S}^c(X)$ is in 1-1 correspondence (via $c_1(L)$) with the set $\mathcal{C}_X$ of characteristic elements.*

As a result, if $H^2(X;\mathbb{Z})$ has no 2-torsion, then we may confuse a *spin^c* structure $\mathfrak{s}$ and its associated determinant line bundle L. If there is 2-torsion, then we can have several different *spin^c* structures $\mathfrak{s}_i$ having the same Chern class $c_1(L_i) = K$.

We will assume for simplicity of the exposition that $H^2(X;\mathbb{Z})$ has no 2-torsion. Let $\mathcal{M}_X^{\delta,g}(K)$ be the moduli space of solutions to certain perturbed monopole equations, where $K \in \mathcal{C}_X$, g is a given metric on X and $\delta \in \Omega^+(X)$ is a perturbation. The moduli space $\mathcal{M}_X^{\delta,g}(K)$ is itself a closed and orientable manifold (for a generic metric g) of dimension $\frac{1}{4}(K^2 - (3\sigma(X) + 2\chi(X)))$. In addition, $\mathcal{M}_X^{\delta,g}(K)$ is a subspace of an infinite-dimensional manifold $\mathcal{B}_K^*$, which is homotopy equivalent to $\mathbb{C}P^\infty$, in particular, implying that $H^*(\mathcal{B}_K^*;\mathbb{Z}) \cong \mathbb{Z}[\mu]$ and $[\mathcal{M}_X^{\delta,g}(K)] \in H_{2m}(\mathcal{B}_K^*;\mathbb{Z})$ is a homology class. (If $H^2(X;\mathbb{Z})$ has 2-torsion, then $\mathcal{B}_K^*$ is homotopic to a disjoint union of $\mathbb{C}P^\infty$s, one for each *spin^c* structure $\mathfrak{s}_i$ corresponding to K.)

Definition 5.37. For X as above, the *Seiberg-Witten invariant* is $SW_X : \mathcal{C}_X \to \mathbb{Z}$ is defined by $SW_X(K) = \left\langle \mu^m, [\mathcal{M}_X^{\delta,g}(K)] \right\rangle$, where dim $\mathcal{M}_X^{\delta,g}(K) = 2m$ and if dim

$\mathcal{M}_X^{\delta,g}(K) < 0$ then $SW_X(K) = 0$. (If dim $\mathcal{M}_X^{\delta,g}(K)$ is odd then $b_2^+(X)$ is even, and we are assuming $b_2^+(X)$ is odd.)

The *Seiberg-Witten invariant* is SW_X is indeed a diffeomorphism invariant: it does not depend on the choices made in its construction.

Definition 5.38. A cohomology class $K \in \mathcal{C}_X \subset H^2(X;\mathbb{Z})$ is a *Seiberg-Witten basic class* if $SW_X(K) \neq 0$, and the set of basic classes denoted by $\mathcal{B}as_X$.

Example 5.39. The following lists all the *basic classes* of the manifolds $E(m)_n$, the logarithmic transforms of the elliptic surfaces:

$$\mathcal{B}as_{E(m)_n} = \{PD(q \cdot f_n) | q \equiv mn - n - 1 (\operatorname{mod} 2), |q| \leq mn - n - 1\}$$

where f_n is the homology class of the multiple fiber of $E(m)_n$

Definition 5.40. A simply connected 4-manifold is said to be of *simple type* if for each $K \in \mathcal{B}as_X$ we have $K^2 = c_1^2(X) = 3\sigma(X) + 2\chi(X)$ (implying that dim $\mathcal{M}_X^{\delta,g}(K) = 0$).

Now we will state some useful results of Seiberg-Witten invariants:

The Seiberg-Witten invariants behave very well under blow-ups ([FS1] for general case):

Theorem 5.41. *The blow-up formula* [GS]. *Let X be a simply connected 4-manifold of simple type with $\mathcal{B}as_X = \{K_i | i = 1, \ldots, s\}$. If $X' = X \# \overline{\mathbb{C}P^2}$ is the blow-up of X and $E \in H^2(X';\mathbb{Z})$ denotes the Poincarè dual of the homology class $e \in H_2(X', \mathbb{Z})$ of the exceptional sphere, then the set of basic classes of X' equals $\{K_i \pm E | i = 1, \ldots, s\}$.*

For Seiberg-Witten behavior under rational blow-downs, we have the following results, [FS2], also see [GS]:

Proposition 5.42. *Let the sphere configuration $C_n \subset X$, and $X_{(n)} = X^\circ \cup B_n$ (where $X^\circ = X - C_n$) be the rational blow-down of X along C_n. Then for every characteristic element $\overline{K} \in \mathcal{C}_{X_{(n)}}$ there is an element $K \in \mathcal{C}_X$ such that $\overline{K}_{|X^\circ} = K_{|X^\circ}$ and $K^2 - \overline{K}^2 = -(n-1)$ (meaning dim $\mathcal{M}_{X_{(n)}}(\overline{K}) =$ dim $\mathcal{M}_X(K)$). The class K is called a lift of $\overline{K}$.*

Theorem 5.43. *Suppose that X and $X_{(n)}$ (as above) are simply connected 4-manifolds. Choose $\overline{K} \in \mathcal{C}_{X_{(n)}}$, and fix a lift $K \in \mathcal{C}_X$ for it. If $K^2 \geq 3\sigma(X) + 2\chi(X)$ (meaning that dim $\mathcal{M}_X(K) \geq 0$), then $SW_{X_{(n)}}(\overline{K}) = SW_X(K)$. Consequently, the Seiberg-Witten invariants of X, SW_X, determine the Seiberg-Witten invariants of the rational blow-down of X, $SW_{X_{(n)}}$.*

Remark 5.44. The theorem above expresses the SW basic classes of $X_{(n)}$ in terms of the SW basic classes of X. Consequently, it tells us which SW basic classes X "pass down" to $X_{(n)}$. It does not, however, provide us a way to reconstruct the SW basic classes of X from those of $X_{(n)}$. In fact, the only basic classes that can "pass down" from X to $X_{(n)}$, are those which when restricted to $\partial X^\circ \cong L(n^2, n-1)$, correspond to an element of order n in $H^2(L(n^2, n-1), \mathbb{Z}) \cong \mathbb{Z}/n^2\mathbb{Z}$.

For complex surfaces S, and a smooth, nonsingular, connected, complex curve $C \subset S$, the standard adjunction formula says that:

$$2g(C) - 2 = [C]^2 - \langle c_1(S), C \rangle \ ,$$

where $g(C)$ is the genus of C [GS]. (Also, look at equation (2.1) for the similar statement for pseudo-holomorphic submanifolds of an almost-complex 4-manifold). The Seiberg-Witten invariants give us the following adjunction formula result for smooth manifolds X:

Theorem 5.45. ***Generalized adjunction formula*** [KM, OzSz], *also see* [GS]. *Assume that $\Sigma \subset X$ is an embedded, oriented, connected surface of genus $g(\Sigma)$ with* *self-intersection* $[\Sigma]^2 \geq 0$ *(and* $[\Sigma] \neq 0$*). Then for every Seiberg-Witten basic class* $K \in \mathcal{B}as_X$ *we have* $2g(\Sigma) - 2 \geq [\Sigma]^2 + |K(\Sigma)|$*. If* X *is of simple type and* $g(\Sigma) > 0$*, the same inequality holds for* $\Sigma \subset X$ *with arbitrary square* $[\Sigma]^2$*.*

There is also further generalization of this result for immersed spheres. We state here a simplified version, where $\dim \mathcal{M}_X^{\delta,g}(K) = 0$:

Theorem 5.46. ***Generalized adjunction formula for immersed spheres*** [FS1]*. Suppose that* X *is an arbitrary smooth* 4*-manifold with* $b_2^+(X) > 1$ *and that* $K \in \mathcal{C}_X$ *with* $SW_X(K) \neq 0$ *and dim* $\mathcal{M}_X(K) = 0$*. If* $x \neq 0 \in H_2(X;\mathbb{Z})$ *is represented by an immersed sphere with p positive double points, then either*

$$2p - 2 \geq x^2 + |x \cdot L|$$

or

$$SW_X(K) = \begin{cases} SW_X(K + 2x), & \text{if } x \cdot K \geq 0 \\ SW_X(K - 2x), & \text{if } x \cdot K \leq 0. \end{cases}$$

The Seiberg-Witten invariants also have interesting behavior if the 4-manifold X is equipped with a symplectic form ω. For example, if a 4-manifold has a symplectic structure then it must be of *simple type*. Additionally, we have the following important results of Taubes:

Theorem 5.47. [Ta1] *If* (X, ω) *is a simply connected symplectic manifold with* $b_2^+(X) > 1$*, then* $SW_X(\pm c_1(X, \omega)) = \pm 1$*.*

Theorem 5.48. [Ta2, Ta4]*, also see* [Ko] *(and* [GS]*, chapter 10, for this simpler statement). Suppose that* (X, ω) *is a symplectic* 4*-manifold with* $b_2^+(X) > 1$ *and* $SW_X(K) \neq 0$ *for a given* $K \in \mathcal{C}_X$*. Assume furthermore that the class* $c = \frac{1}{2}(K - c_1(X, \omega))$ *is nonzero in* $H^2(X;\mathbb{Z})$*. Then for a generic compatible almost-complex structure* J *on* X*, the class* $PD(c) \in H_2(X;\mathbb{Z})$ *can be represented by a pseudo-holomorphic submanifold (not necessarily connected).*

From the above result, one can also conclude the following:

Theorem 5.49. [Ta4, Ta3, Ko], *also see* [GS]. *If X is a minimal symplectic* 4-*manifold (i.e. does not contain symplectic spheres with self-intersection* (-1)*) with* $b_2^+(X) > 1$, *then* $c_1^2(X,\omega) \geq 0$.

From the above two results and the generalized adjunction formula, we can conclude the following:

Corollary 5.50. [Ta3], *also see* [GS]. *If (X,ω) is a symplectic* 4-*manifold with* $c_1^2(X,\omega) \leq -1$, *then for a generic compatible almost-complex structure J on X, there exists a J-holomorphic sphere of self-intersection* (-1).

Proof. From Theorem 5.49 it follows that if $c_1^2(X,\omega) \leq -1$, then there exists a symplectic sphere, $\Sigma \in X$, with $[\Sigma]^2 = -1$. However, since for the homology class $\pm[\Sigma]$ we have $c_1(X,\omega) \cdot PD([\Sigma]) = 1$ and $SW_X(c_1(X,\omega) + 2PD([\Sigma])) \neq 0$, meaning that $c_1(X,\omega) + 2PD([\Sigma]) \in \mathcal{B}as_X$, then from Theorem 5.48 we have that the homology class $[\Sigma]$ can be represented by a pseudo-holomorphic submanifold. Finally, the generalized adjunction formula forces the pseudo-holomorphic submanifold to be a sphere. $\square$

5.3. **Proof of Theorem 5.6.**

5.3.1. *Step 1.* As a first step in proving Theorem 5.6, we will prove Proposition 5.2, that is, we will show that there exists a sphere, Σ_{-1} of self-intersection (-1) which intersects the spheres of the C_n configuration, positively and transversally, in the rational blow-up of X.

We begin by assuming that for a given symplectic 4-manifold (X,ω), with conditions as stated in the Proposition 5.2, there is a symplectic embedding $B_n \hookrightarrow (X,\omega)$. This embedding is in the sense of Theorem 3.8, meaning there is a Lagrangian core

$\mathcal{L}_{n,1}$ in (X,ω), whose neighborhood is the rational homology ball B_n. It follows, according to Theorem 3.8, we can perform the symplectic rational blow-up procedure, and obtain a new symplectic manifold (X',ω') which contains a symplectic copy of a C_n configuration of symplectic spheres. Since we assumed that $n \geq c_1^2(X,\omega)+2$, and since $c_1^2(X',\omega') = c_1^2(X,\omega)-(n-1)$, we have $c_1^2(X',\omega') \leq -1$. As a consequence of Corollary 5.50, for a generic compatible almost-complex structure J_ϵ on X', there exists a J_ϵ-holomorphic sphere, Σ_ϵ^{-1} with self-intersection number (-1).

In order to force only positive intersections between the spheres of the C_n configuration and a sphere of self-intersection (-1), Σ_{-1} (derived from Σ_ϵ^{-1} as a consequence of Proposition 5.61), we need to make the spheres of the C_n configuration pseudo-holomorphic:

Lemma 5.51. *With X' as above, there exists an ω-compatible almost-complex structure J on X' such that all of the spheres in the C_n configuration are J-holomorphic.*

Proof. First, we label the spheres of C_n with $S_1, S_2, S_3, \ldots, S_{n-1}$, as before in Figure 25. Let the points $a_i = S_i \cap S_{i+1}$ be the points in the intersection of the spheres of C_n. Let N_{a_i} be small Darboux neighborhoods around those points, such that

$$E = \bigcup_{i=1}^{n-1} S_i - \bigcup_{i=1}^{n-2} (N_{a_i} \cap (S_i \cup S_{i+1})) \tag{5.6}$$

is a symplectic submanifold consisting of $(n-1)$ connected components. Then, we can choose an ω-compatible almost-complex structure J on X' such that all the connected components of the submanifold E are J-holomorphic submanifolds.

We can extend this almost-complex structure J across the neighborhoods of the intersection points N_{a_i} as follows: First, the results of [McPo] imply that for the symplectic spheres in C_n configuration, which intersect transversally and positively, can always be isotoped in such a way that they intersect orthogonally (with respect to the symplectic structure) while remaining symplectic. Second, we use the following technical local result, which is a version of McDuff's result in [Mc]:

Lemma 5.52. *Let π_1 and π_2 be two orthogonal planes through $\{0\}$ in $\mathbb{R}^4$ which intersect with positive orientation and are symplectic with respect to the standard linear symplectic form ω_0. Then there is a linear ω_0-compatible J which preserves these planes.*

Proof. We can choose a basis (e_1, e_2) for $\pi_1 \subset \mathbb{R}^4$ and a basis (e_3, e_4) for $\pi_2 \subset \mathbb{R}^4$, such that $\omega_0(e_1, e_2) = 1$ and $\omega_0(e_3, e_4) = 1$ and $\pi_1^{\perp} = \pi_2$ (with respect to ω_0). Then we simply choose J to be such that $J(e_1) = e_2$ and $J(e_3) = (e_4)$. $\square$

Since after (possibly) isotoping the symplectic spheres of C_n, the intersections of the spheres are orthogonal, in a local Darboux neighborhood, N_{a_i}, they can be modeled by two orthogonal planes through $\{0\}$ in $\mathbb{R}^4$. Therefore, Lemma 5.52 implies that we can choose an ω-compatible almost-complex structure J on X' such that the symplectic spheres of C_n are also J-holomorphic spheres. $\square$

Remark 5.53. McDuff's result [Mc], says that if the planes π_1 and π_2 intersect positively and transversally then there exists an ω-tame almost-complex structure J preserving the planes. This is not enough for our purposes, since in the next step, using Gromov compactness we will consider a sequence of almost-complex structures from $J_\epsilon \to J$, and since J_ϵ is required to be ω-compatible by Taubes' theorem, we need J to be ω-compatible as well.

Proposition 5.54. *Let X' be the rational blow-up of X, as above, then there exists a J-holomorphic sphere of self-intersection (-1) in X', Σ_{-1}, with J the almost-complex structure from Lemma 5.51.*

Proof. To show the existence of this J-holomorphic sphere, Σ_{-1} we will use Gromov compactness to find a sequence of almost-complex structures, under which the

J_ϵ-holomorphic sphere Σ_ϵ^{-1} will converge to a multicurve, (or a cusp-curve) with (potentially) some "bubbles". One of the components of the multicurve will be a J-holomorphic sphere of self-intersection (-1), Σ_{-1}. First, we state the definition and properties of a multicurve, convergence of almost-complex structures and Gromov compactness. Let (M,ω) be a compact symplectic manifold:

Definition 5.55. [MS1] A *multicurve* (or *cusp-curve*) C is a connected union

$$C = C^1 \cup C^2 \cup \cdots \cup C^N \tag{5.7}$$

of J-holomorphic spheres C^j, which are called components. Each component is parameterized by a smooth nonconstant J-holomorphic map $u^j : \mathbb{C}P^1 \to M$, which is not required to be simple. The multicurve is denoted by $u = (u^1, \ldots, u^N)$.

Definition 5.56. [MS1] A sequence of J-holomorphic curves $u_\nu : \mathbb{C}P^1$ is said to **converge weakly** to a multicurve $u = (u^1, \ldots, u^N)$ if the following holds:

(1) For every $j \leq N$, there exists a sequence $\phi_\nu^j : \mathbb{C}P^1 \to \mathbb{C}P^1$ of fractional linear transformations and a finite set $X^j \subset \mathbb{C}P^1$ such that $u_\nu \circ \phi_\nu^j$ converges to u^j uniformly with all derivates on compact subsets of $\mathbb{C}P^1 - X^j$.
(2) There exists a sequence of orientation preserving (but not holomorphic) diffeomorphisms $f_\nu : \mathbb{C}P^1 \to \mathbb{C}P^1$ such that $u_\nu \circ f_\nu$ converges in the C^0-topology to a parametrization $v : \mathbb{C}P^1 \to M$ of the multicurve $u = (u^1, \ldots, u^N)$.

It follows [MS1], that for ν sufficiently large, that the map $u_\nu : \mathbb{C}P^1 \to M$ is homotopic to:

$$u^1 \# u^2 \# \cdots \# u^N : \mathbb{C}P^1 \to M\,. \tag{5.8}$$

In particular if $A_\nu, A^j \in H_2(M,\mathbb{Z})$ are the homology classes of u_ν and u^j respectively, then we have:

$$c_1(M) \cdot A_\nu = \sum_{j=1}^{N} c_1(M) \cdot A^j \,. \tag{5.9}$$

Finally, we can state Gromov's compactness theorem [Gr], as it appears in [MS1]:

Theorem 5.57. ***(Gromov's compactness)*** *Assume M is compact, and let $J_\nu \in \mathcal{J}_\tau(M,\omega)$ be a sequence of ω-tame almost complex structures which converge to J in the C^∞-topology. Then any sequence $u_\nu : \mathbb{C}P^1 \to M$ of J_ν-holomorphic spheres with $sup_\nu E(u_\nu) < \infty$ has a subsequence which converges weakly to a (possible reducible) J-holomorphic multicurve $u = (u^1, \ldots, u^N)$.*

Additionally, specifically for symplectic manifolds of dimension 4, we have the following adjunction formula:

Theorem 5.58. *Adjunction Formula (*[MS3] *App. E). Let (M, J) be an almost-complex 4-manifold, (Σ, J) be a closed Riemann surface, not necessarily connected, and $u : \Sigma \to M$ be a simple J-holomorphic curve. Denote $A \in H_2(M;\mathbb{Z})$ the homology represented by u. Then*

$$2\delta(u) \le A \cdot A - c_1(M) \cdot A + \chi(\Sigma) \tag{5.10}$$

with equality if and only if u is an immersion and all self-intersections are transverse.

In the above, "simple" means not multiply covered and $\delta(u)$ is the number of self-intersections of u:

$$\delta(u) := \frac{1}{2}\#\left\{(z_0, z_1) \in \Sigma \times \Sigma | u(z_0) = u(z_1), z_0 \neq z_1\right\} \tag{5.11}$$

Additionally, McDuff also proved the following corollary to the theorem above:

Corollary 5.59. *(*[MS3] *App. E). Let M, Σ, u and A be as in Theorem 5.58. Then*

$$A \cdot A - c_1(M) \cdot A + \chi(\Sigma) \geq 0 \tag{5.12}$$

with equality if and only if u is an embedding.

In [MS3], McDuff proves Theorem 5.58 by showing that in dimension 4, a homology class $A \in H_2(M, \mathbb{Z})$ which is represented by a simple J-holomorphic curve, $u : \Sigma \to M$, can always be represented by an immersed J'-holomorphic curve $v : \Sigma \to M$, with transverse self-intersections. The curves u and v are C^1-close, and the almost-complex structures J and J' are C^1-close as well. This is shown using a strong theorem of Micallef-White [MW] which states that a singularity of a J-holomorphic curve is equivalent to a singularity of a holomorphic curve, up to a C^1-diffeomorphism. As a result, Li in [Li] made the following observation:

Lemma 5.60. *If a homology class $A \in H_2(M; \mathbb{Z})$, for a 4-dimensional symplectic manifold M, is represented by a simple J-holomorphic curve $u : \Sigma \to M$ for some ω-tamed almost-complex structure J, then A is represented by an embedded symplectic surface.*

In our case, the spheres of the C_n configuration are J-holomorphic, whereas the sphere with self-intersection (-1), Σ_ϵ^{-1}, is J_ϵ-holomorphic. So by Gromov's compactness theorem, we can take a sequence of almost-complex structures $J_\epsilon \to J$, such that there will exist a subsequence under which the J_ϵ-holomorphic sphere Σ_ϵ^{-1} will converge to some multicurve $u = (u^1, \dots, u^N)$. Since the u^i's can be multiply covered (multiplicity m_i), we will write v^i for the underlying simple J-holomorphic curve, giving us $[u^i] = m_i[v^i]$ as homology classes in $H_2(X'; \mathbb{Z})$. Also, we have:

$$[\Sigma_\epsilon^{-1}] = m_1[v^1] + m_2[v^2] + \cdots + m_N[v^N] \tag{5.13}$$

in $H_2(X';\mathbb{Z})$. Next, in Proposition 5.61, our goal is to show that one of the v^i's is indeed an embedded J-holomorphic sphere of self-intersection (-1) in X'.

Proposition 5.61. *Let Σ_ϵ^{-1} and v^i, $i \in \{1, \ldots, N\}$, as in the above paragraph. Then for at least one i, the simple J-holomorphic curve v^i is an embedded sphere with self-intersection (-1).*

Proof. If $N = 1$, then $m_1 = 1$ and $c_1(X') \cdot [v^1] = 1$, applying the inequality (5.12) for v^1, we have that $[v^1]^2 \geq -1$. If $[v^1]^2 = -1$, then by Corollary 5.59 it must be an embedding. If $[v^1]^2 = k \geq 0$, then by Lemma 5.60, there exists an embedded symplectic surface v_S^1, with $[u^1] = [v_S^1]$, for which we have $-\chi(v_S^1) = [v_S^1]^2 - c_1(X') \cdot [v_S^1] = k - 1$. However, if this is the case then this violates the generalized adjunction formula, since we would then have $k - 1 \geq k + |c_1(X') \cdot [v_S^1]|$, which cannot occur.

We will prove this proposition for general N with an inductive combinatorial argument using Corollary 5.59, Lemma 5.60, the adjunction formula for embedded symplectic surfaces, as well as the generalized adjunction formula (Theorem 5.45). First, (although not strictly necessary for the proof), we will prove the proposition for $N = 2$, and make a slightly stronger assumption for the initial inductive case, in order to go to the general inductive step in a less cumbersome manner. If $N = 2$, then we have:

$$[\Sigma_\epsilon^{-1}] = m_1[v^1] + m_2[v^2] \qquad m_1 c_1(X') \cdot [v^1] + m_2 c_1(X') \cdot [v^2] = 1 \tag{5.14}$$

Case 1: Assume $[v^1]^2 = 2k \geq 0$, then by inequality (5.12), we have: $c_1(X') \cdot [v^1] \leq 2 + 2k$, therefore:

$$\begin{aligned}
\text{If}\, c_1(X') \cdot [v^1] = 2 + 2k &\Rightarrow v^1 \text{ must be embedded} \\
\text{If}\, c_1(X') \cdot [v^1] = 2k &\Rightarrow \exists v_S^1 \text{ s.t. } -\chi(v_S^1) = 0 \\
\text{If}\, c_1(X') \cdot [v^1] = 2k - 2 &\Rightarrow \exists v_S^1 \text{ s.t. } -\chi(v_S^1) = 2 \\
\vdots \quad &\Rightarrow \quad \vdots \\
\text{If}\, c_1(X') \cdot [v^1] = 2 &\Rightarrow \exists v_S^1 \text{ s.t. } -\chi(v_S^1) = 2k - 2
\end{aligned}$$

where v_S^1 is an embedded symplectic surface such that $[v^1] = [v_S^1]$. This forces $c_1(X') \cdot [v^1] \leq 0$, since if $2 \leq c_1(X') \cdot [v^1] \leq 2 + 2k$, then the embedded surface v_S^1 fails to satisfy the generalized adjunction formula (Theorem 5.45). Also, note that $c_1(X') \cdot [v^1]$ must be an even integer. Next, (5.14) together with $c_1(X') \cdot [v^1] \leq 0$ imply that $c_1(X') \cdot [v^2] \geq 1$. If we apply (5.12) to v^2, we get: $[v^2]^2 \geq -1$. Thus, if $[v^2]^2 = -1$, then $c_1(X') \cdot [v^2] = 1$ and by Corollary 5.59 v^2 is an embedding, if not then $[v^2]^2 = l \geq 0$ and by (5.12) we get $1 \leq c_1(X') \cdot [v^2] \leq l + 2$, so:

$$\begin{aligned}
\text{If}\, c_1(X') \cdot [v^2] = l + 2 &\Rightarrow \exists v_S^2 \text{ s.t. } -\chi(v_S^2) = -2 \\
\text{If}\, c_1(X') \cdot [v^2] = l &\Rightarrow \exists v_S^2 \text{ s.t. } -\chi(v_S^2) = 0 \\
\text{If}\, c_1(X') \cdot [v^2] = l - 2 &\Rightarrow \exists v_S^2 \text{ s.t. } -\chi(v_S^2) = 2 \\
\vdots \quad &\Rightarrow \quad \vdots \\
\text{If}\, c_1(X') \cdot [v^2] = 1 &\Rightarrow \exists v_S^2 \text{ s.t. } -\chi(v_S^2) = l - 1 (\text{if}\, l \,\text{is even})
\end{aligned}$$

where v_S^2 is an embedded symplectic surface such that $[v^2] = [v_S^2]$. Here, we must have $[v^2]^2 = -1$, since all the cases where $[v^2]^2 = l \geq 0$ and $1 \leq c_1(X') \cdot [v^2] \leq l + 2$, cannot occur because applying the generalized adjunction formula (Theorem 5.45) would result in a contradiction. Consequently, if $[v^1]^2 = 2k \geq 0$, then we must have $[v^2]^2 = -1$ and v^2 must be an embedded sphere.

Case 2: Assume $[v^1]^2 = 2k+1 \geq 0$. If we apply the inequality (5.12) to v^1, then we have $c_1(X') \cdot [v^1] \leq 3+2k$. However, just as in *Case 2*, if $1 \leq c_1(X') \cdot [v^1] \leq 3+2k$, then there would exist an embedded symplectic surface v^1_S, with $[v^1] = [v^1_S]$, such that applying the generalized adjunction formula (Theorem 5.45) for v^1_S would result in a contradiction. Also, as before, we again observe that the integer $[v^1]^2 - c_1(X') \cdot [v^1]$ must be even, thus we have $c_1(X') \cdot [v^1] \leq -1$.

We proceed as before in *Case 1*, and $c_1(X') \cdot [v^1] \leq -1$ together with equation (5.14), imply that $1 \leq c_1(X') \cdot [v^2]$. Therefore, by the same steps as in *Case 1*, if $[v^1]^2 = 2k+1$, then we must have $[v^2]^2 = -1$, and v^2 must be an embedded sphere.

We can switch the roles of v^1 and v^2, in the above cases, which implies that if $[v^2]^2 = k \geq 0$ then $[v^1]^2 = -1$ and v^1 must be an embedded sphere. Therefore, we are left with case:

Case 3: Assume both $[v^1]^2 \leq -1$ and $[v^2]^2 \leq -1$. We can again apply inequalities (5.12) to v^1 and v^2, multiplying the first by m_1 and the second one by m_2, adding them together, and using (5.14), we get:

$$1 - 2m_1 - 2m_2 \leq m_1[v^1]^2 + m_2[v^2]^2\,,$$

implying that both $[v^1]^2$ and $[v^2]^2$ can't be ≤ 2. Therefore, we are left with a finite number of possibilities: Either $[v^1]^2 = -1$ and $[v^2]^2 = -k \leq -1$ (satisfying inequality (5.3.1)) or the same with roles of v^1 and v^2 switched. In this case we have the following:

$$\left.\begin{array}{l} [v^1]^2 = -1 \\ {[v^2]}^2 = -k \leq -1 \end{array}\right\} \implies \begin{array}{l} c_1(X') \cdot [v^1] \leq 1 \\ c_1(X') \cdot [v^2] \leq 2-k. \end{array}$$

If $k = 1$ then, (5.14) implies that at least one of $c_1(X') \cdot [v^i]$ must be 1, in turn implying that either v^1 or v^2 is an embedded sphere with self-intersection (-1). If $k > 1$, then again because of (5.14), we must have $c_1(X') \cdot [v^1] = 1$ implying that v^1 is an embedded sphere with self-intersection (-1). If roles of v^1 and v^2 are switched, with $[v^1]^2 = -k \leq -2$ and $[v^2]^2 = -1$, we would have v^2 be an embedded sphere with self-intersection (-1).

This covers all the possibilities of the values for $[v^1]^2$ and $[v^2]^2$ with $N = 2$, and in each case at least one of v^1 or v^2 is an embedded sphere with self-intersection (-1). We observe that if we replace the heavily used equation (5.14), by:

$$m_1 c_1(X') \cdot [v^1] + m_2 c_1(X') \cdot [v^2] = m \geq 1 \tag{5.15}$$

then everything in the *Cases 1-3* would proceed in the same way. In *Case 1*, whenever we have $c_1(X') \cdot [v^1] \leq 0$, we can still use equation (5.15) to conclude that $c_1(X') \cdot [v^2] \geq 1$, and everything would proceed in the same way. In *Case 2*, whenever we have $c_1(X') \cdot [v^1] \leq -1$, again we can still use equation (5.15) to conclude that $c_1(X') \cdot [v^2] \geq 1$. Likewise in *Case 3*, $m \geq 1$ in (5.15) is all that is needed to reach the desired conclusion.

Consequently, for a configuration of J-holomorphic curves $m_1[v^1]+m_2[v^2]$, with the condition (5.15), at least one of the curves v^1 and v^2 must be an embedded sphere with self-intersection (-1). We make an induction assumption, that if we have a configuration of J-holomorphic curves $m_1[v^1] + m_2[v^2] + \cdots + m_{N-1}[v^{N-1}]$, with the condition:

$$m_1 c_1(X') \cdot [v^1] + m_2 c_1(X') \cdot [v^2] + \cdots + m_{N-1} c_1(X') \cdot [v^{N-1}] = m \geq 1 \tag{5.16}$$

then one of the v^is, $1 \leq i \leq N-1$, is an embedded sphere with self-intersection (-1). We will show that if we have a configuration of J-holomorphic curves $m_1[v^1] +$

$m_2[v^2] + \cdots + m_N[v^N]$, with the condition:

$$(5.17) \qquad m_1c_1(X') \cdot [v^1] + m_2c_1(X') \cdot [v^2] + \cdots + m_Nc_1(X') \cdot [v^N] = m' \geq 1$$

then one of the v^is, $1 \leq i \leq N$, is an embedded sphere with self-intersection (-1).

Case 1': Assume $[v^N]^2 = 2k \geq 0$. Then by (5.12), we have $c_1(X') \cdot [v^N] \leq 2k+2$. However, as in *Case 1* from $N = 2$, by Lemma 5.60 the existence of a smooth symplectic surface v^N_S, with $[v^N] = [v^N_S]$, together with the generalized adjunction formula (Theorem 5.45) imply that in fact $c_1(X') \cdot [v^N] \leq 0$. Combining this with (5.17), we get:

$$\begin{aligned} & m' - m_1c_1(X') \cdot [v^1] - \cdots - m_{N-1}c_1(X') \cdot [v^{N-1}] = m_Nc_1(X') \cdot [v^N] \leq 0 \\ \Rightarrow \quad & m_1c_1(X') \cdot [v^1] + \cdots + m_{N-1}c_1(X') \cdot [v^{N-1}] \geq m' \geq 1 \end{aligned}$$

which according to the induction hypothesis implies that at least one v^is, $1 \leq i \leq N-1$, is an embedded sphere with self-intersection (-1).

Case 2': Assume $[v^N]^2 = 2k+1 \geq 1$. Again, by (5.12), we have $c_1(X') \cdot [v^N] \leq 2k+3$. However, as in *Case 2* from $N = 2$, we have $c_1(X') \cdot [v^N] \leq -1$, and combining this with (5.17), we get:

$$\begin{aligned} & m' - m_1c_1(X') \cdot [v^1] - \cdots - m_{N-1}c_1(X') \cdot [v^{N-1}] = m_Nc_1(X') \cdot [v^N] \leq -m_N \\ \Rightarrow \quad & m_1c_1(X') \cdot [v^1] + \cdots + m_{N-1}c_1(X') \cdot [v^{N-1}] \geq m' + m_N \geq 1 \end{aligned}$$

which again according to the induction hypothesis implies that at least one v^is, $1 \leq i \leq N-1$, is an embedded sphere with self-intersection (-1).

Case 3': Assume $[v_N]^2 = -1$. Applying (5.12) to v^N, we get $c_1(X') \cdot v_N \leq 1$. If $c_1(X') \cdot v_N = 1$, then v_N is an embedded sphere. Otherwise, $c_1(X') \cdot v_N \leq -1$, and as in *Case 2'*, we have:

$$(5.18) \qquad m_1c_1(X') \cdot [v^1] + \cdots + m_{N-1}c_1(X') \cdot [v^{N-1}] \geq m' + m_N \geq 1$$

which by the induction hypothesis would imply that at least one of the v^is, for $1 \leq i \leq N-1$, is an embedded sphere with self-intersection (-1).

Case 4' Assume $[v_N]^2 = -2$. Again, applying (5.12) to v^N, we get $c_1(X') \cdot v_N \leq 0$, meaning we have:

$$m_1 c_1(X') \cdot [v^1] + \cdots + m_{N-1} c_1(X') \cdot [v^{N-1}] \geq m' \geq 1 \tag{5.19}$$

which by the induction hypothesis again would imply that at least one of the v^is, for $1 \leq i \leq N-1$, is an embedded sphere with self-intersection (-1).

Applying *Cases 1'-4'* to every v^i, $1 \leq i \leq N-1$, and applying the induction hypothesis each time, gives us that for all the instances where $[v^i]^2 \geq -2$, for any $1 \leq i \leq N$, we will have a v^j, for at least one $1 \leq j \leq N$, that is an embedded sphere with self-intersection (-1). Therefore, the only remaining cases is when $[v^i]^2 = -k_i \leq -3$ for all $1 \leq i \leq N$. In which case, we would have $c_1(X') \cdot [v^i] \leq 2 - k_i \leq -1$ for all $1 \leq i \leq N$, which would violate the assumption (5.17). This concludes the induction argument. As a result, when $m' = 1$, this is the case of the Proposition 5.61. $\square$

As a result of Propostion 5.61, we now have a J-holomorphic embedded sphere of self-intersection (-1) in (X', ω'), which we will name Σ_{-1}, along with a C_n configuration of J-holomorphic spheres. This proves Proposition 5.54 $\square$

An important feature of J-holomorphic curves, proven by McDuff [MS3], is that their intersections are always positive. In fact, we can always perturb a set of J-holomorphic curves and obtain embedded symplectic surfaces intersecting positively and transversally. Li-Usher in [LU], develop McDuff's techniques further, in order to perturb several J-holomorphic curves at once, and obtain the following result:

Lemma 5.62. [LU] *Any set of distinct J-holomorphic curves $C_0, \ldots, C_m$ can be perturbed to symplectic surfaces $C'_0, \ldots, C'_m$ whose intersections are all transverse and positive, with $C'_i \cap C'_j \cap C'_k = \varnothing$ when i, j, k are all distinct. Furthermore, there* *is an almost-complex structure J' arbitrarily C^1-close to J such that the C_i' are J'-holomorphic.*

This is shown by modeling a neighborhood around each intersection point or singularity with holomorphic coordinates, and then slightly perturbing each branch.

Proposition 5.2 is now a direct consequence of Lemma 5.62, Lemma 5.51 and Proposition 5.54.

5.3.2. *Step 2.* In this next step of our proof of Theorem 5.6, we use Σ_{-1} to construct a homology class γ and compute $c_1(X) \cdot \gamma$ in terms of the intersection numbers of Σ_{-1} with the spheres of the C_n configuration.

We begin by rationally blowing down the C_n configuration in (X', ω') symplectically. We can do so by the definition of the symplectic rational blow-down of Symington in [Sy3]. We choose a neighborhood $(N(C_n), \omega'|_{N(C_n)})$ of the spheres in C_n, such that $\partial(N(C_n)) \cap \Sigma_{-1} \cong S^1$, $N(C_n) \cap \Sigma_{-1} \cong D^2$ and $(X' \backslash N(C_n)) \cap \Sigma_{-1} \cong D^2$. We denote this rational blow-down of (X', ω') as $(\tilde{X}', \tilde{\omega}')$. We observe that the symplectic manifolds (X, ω) and $(\tilde{X}', \tilde{\omega}')$ differ only by the volume of the rational homology ball B_n. This is due to the non-uniqueness of the symplectic rational blow-up operation, in terms of the symplectic volume of the B_ns. This also implies that the symplectic rational blow-down and the symplectic rational blow-up are not strictly inverse operations. However, $(\tilde{X}', \tilde{\omega}')$ still has the properties that (X, ω) does: $[c_1(\tilde{X}', \tilde{\omega}')] = -[\tilde{\omega}']$, $b_2^+(\tilde{X}') > 1$, $\mathcal{B}as_X = \left\{\pm c_1(\tilde{X}', \tilde{\omega}')\right\}$ and $n \geq c_1^2(\tilde{X}', \tilde{\omega}') + 2$. Therefore, for the remainder of the proof, we will abuse notation and write (X, ω) for $(\tilde{X}', \tilde{\omega}')$.

Back up in X', we can split up rational homology classes as follows:

$$\begin{array}{rcccc} H_2(X'; \mathbb{Q}) & = & H_2(C_n; \mathbb{Q}) & \oplus & H_2(X' \backslash C_n; \mathbb{Q}) \\ \Sigma_{-1} & = & a & + & b \\ PD(c_1(X', \omega')) & = & c & + & d \end{array}$$

Since we have $c_1(X', \omega') \cdot [\Sigma_{-1}] = 1$, then we have $1 = a \cdot c + b \cdot d$.

Let D be a 2-disk defined by:

$$D = (X'\backslash N(C_n)) \cap \Sigma_{-1} \subset X. \tag{5.20}$$

Observe that $D \subset X$, since by definition $X \cong (X'\backslash N(C_n)) \cup B_n$. Also, since $\partial D \subset \partial B_n$ and $H_1(B_n;\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$, then $n\partial D \cong 0 \in H_1(B_n;\mathbb{Q})$. Back down in X, we can now define the class $\gamma \in H_2(X;\mathbb{Q})$ by:

$$\gamma = nD + e^2 \tag{5.21}$$

where e^2 is just a 2-cell in $B_n \subset X$ for which $\partial(e^2) = n\partial D$. Since, $c_1(X', \omega') \cdot [\Sigma_{-1}] = a \cdot c + b \cdot d$ and $H_2(B_n;\mathbb{Q})$ is trivial, we have:

$$c_1(X, \omega) \cdot \gamma = nb \cdot d\,. \tag{5.22}$$

Our goal is to compute $c_1(X, \omega) \cdot \gamma$ explicitly in terms of the intersections of the sphere Σ_{-1} with the spheres of C_n. Next, in ***Step 3*** we will show that whenever $c_1(X, \omega) \cdot \gamma > 0$ then we also have $\omega \cdot \gamma > 0$, thus contradicting the condition $[c_1(X, \omega)] = -[\omega]$. In ***Step 4*** we will show that the intersection configurations of Σ_{-1} with C_n yielding $c_1(X, \omega) \cdot \gamma \leq 0$ will also produce a contradiction.

In order to compute $c_1(X, \omega) \cdot \gamma$, all we need to compute is $a \cdot c$, since $nb \cdot d = n(1 - a \cdot c)$, which is fairly standard. Recall, we denote the spheres of the C_n configuration by $S_1, S_2, S_3, \ldots, S_{n-1}$, with $[S_1]^2 = -n-2$ and $[S_i]^2 = -2$ for $2 \leq i \leq n-1$. Thus, we may denote the basis of $H_2(C_n;\mathbb{Q})$ by $[S_1], [S_2], [S_3], \ldots, [S_{n-1}]$. As a result "$a$", the homology class of Σ_{-1} lying in $H_2(C_n;\mathbb{Q})$, may be expressed as:

$$a = a_1[S_1] + a_2[S_2] + a_3[S_3] + \cdots + a_{n-1}[S_{n-1}] \tag{5.23}$$

where $a_i \in \mathbb{Q}$. Next, let I_j be the intersection numbers of $[\Sigma_{-1}]$ and $[S_j]$:

$$\begin{aligned}
[\Sigma_{-1}] \cdot [S_1] &= I_1 \\
[\Sigma_{-1}] \cdot [S_2] &= I_2 \\
[\Sigma_{-1}] \cdot [S_3] &= I_3 \\
\vdots &= \vdots \\
[\Sigma_{-1}] \cdot [S_{n-1}] &= I_{n-1}\,.
\end{aligned}$$

(Note, we have $\alpha_j = I_j$ (see Definition 5.3), since the intersections of the sphere Σ_{-1} with the spheres S_j are positive and transverse.) In order to express the a_i in terms of the intersection numbers I_j, we need to solve the following linear system:

$$\begin{aligned}
(a_1[S_1] + a_2[S_2] + a_3[S_3] + \cdots + a_{n-1}[S_{n-1}]) \cdot [S_1] &= I_1 \\
(a_1[S_1] + a_2[S_2] + a_3[S_3] + \cdots + a_{n-1}[S_{n-1}]) \cdot [S_2] &= I_2 \\
(a_1[S_1] + a_2[S_2] + a_3[S_3] + \cdots + a_{n-1}[S_{n-1}]) \cdot [S_3] &= I_3 \\
\vdots &= \vdots \\
(a_1[S_1] + a_2[S_2] + a_3[S_3] + \cdots + a_{n-1}[S_{n-1}]) \cdot [S_{n-1}] &= I_{n-1}\,.
\end{aligned}$$

Next, we can express "c", the homology class of $PD(c_1(X', \omega'))$ lying in $H_2(C_n; \mathbb{Q})$, in terms of the basis $[S_1], [S_2], [S_3], \ldots, [S_{n-1}]$:

$$c = c_1[S_1] + c_2[S_2] + c_3[S_3] + \cdots + c_{n-1}[S_{n-1}] \tag{5.24}$$

where the $c_i \in \mathbb{Q}$. Since the S_i are symplectic spheres, we have the following:

$$\begin{aligned} c_1(X') \cdot [S_1] &= -n \\ c_1(X') \cdot [S_2] &= 0 \\ c_1(X') \cdot [S_3] &= 0 \\ \vdots &= \vdots \\ c_1(X') \cdot [S_{n-1}] &= 0\,. \end{aligned}$$

As a result, the quantity $a \cdot c$ is the dot product of the following two vectors in $H_2(C_n; \mathbb{Q})$:

$$[a_1, a_2, a_3, \ldots, a_{n-1}] \tag{5.25}$$

and

$$[-n, 0, 0, \ldots, 0]\,. \tag{5.26}$$

Consequently, we only have to compute a_1 in terms of the intersection numbers I_j, which corresponds to the first row of the inverse of the $H_2(C_n; \mathbb{Z})$ intersection matrix, giving us:

$$a_1 = \frac{-n+1}{n^2} I_1 + \frac{-n+2}{n^2} I_2 + \cdots + \frac{-2}{n^2} I_{n-2} + \frac{-1}{n^2} I_{n-1}\,. \tag{5.27}$$

Since $a \cdot c = a_1 \cdot n$ and $c_1(X, \omega) \cdot \gamma = n(1 - a \cdot c)$, we finally get:

$$c_1(X, \omega) \cdot \gamma = n - I_{n-1} - 2I_{n-2} - 3I_{n-3} - \cdots - (n-2)I_2 - (n-1)I_1\,. \tag{5.28}$$

Note, that since $\alpha_j = I_j$, then we have shown that if the symplectic embedding of $B_n \hookrightarrow X$ is of type $\langle \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1} \rangle$, then there is a class γ, such that $c_1(X) \cdot \gamma$ is given by (5.28).

5.3.3. *Step 3.* In this step we will show that if $c_1(X,\omega)\cdot\gamma > 0$, then we must also have $\omega\cdot\gamma > 0$, thus violating the $[c_1(X,\omega)] = -[\omega]$ condition of (X,ω). This will eliminate the possibility of embeddings $B_n \hookrightarrow X$ of type $\mathcal{A}_1 \subset \mathcal{A}$, where $\mathcal{A}_1$ is the set of $(n-1)-$tuples $\langle \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}\rangle$ satisfying the inequality (5.29) (with $\alpha_j = I_j$).

If $c_1(X)\cdot\gamma > 0$, we have:

$$n - I_{n-1} - 2I_{n-2} - 3I_{n-3} - \cdots - (n-2)I_2 - (n-1)I_1 > 0. \tag{5.29}$$

First, we will use the following lemma to rule out some cases.

Lemma 5.63. *Let Σ and S be embedded spheres in a smooth 4-manifold M with $b_2^+(M) > 1$, such that $[\Sigma]^2 = -1$ and $[S]^2 = -2$. Assume $[\Sigma]\cdot[S] = k \geq 1$, then we must have $k = 1$.*

Proof. The proof will follow from the following proposition:

Proposition 5.64. [FM] *Let M be an oriented 4-manifold and $S^2 \subset M$ be an embedded sphere with $\alpha \in H^2(M;\mathbb{Z})$ the cohomology class dual to S^2. If $\alpha^2 = -1$ or -2, there is an orientation preserving self-diffeomorphism φ of M such that $\varphi^* = R_\alpha$, where:*

$$R_\alpha(x) = x + 2(x\cdot\alpha)\alpha \tag{5.30}$$

if $\alpha^2 = -1$ and

$$R_\alpha(x) = x + (x\cdot\alpha)\alpha \tag{5.31}$$

if $\alpha^2 = -2$. (Note, in both cases $R_\alpha = -\alpha$ and $R_\alpha^2 = Id$, hence often referred to as the reflection automorphism.)

As a result of this proposition, the spheres Σ and S will induce orientation preserving diffeomorphisms on M, corresponding to the following reflection automorphisms on $H_2(M;\mathbb{Z})$:

$$R_\Sigma(x) = x + 2(x \cdot [\Sigma])[\Sigma] \tag{5.32}$$

$$R_S(x) = x + (x \cdot [S])[S]\,. \tag{5.33}$$

We begin with applying R_S to $x = [\Sigma]$:

$$R_S([\Sigma]) = [\Sigma] + k[S]\,. \tag{5.34}$$

Next, we apply R_Σ to $x = [\Sigma] + k[S]$:

$$R_\Sigma([\Sigma] + k[S]) = (2k^2 - 1)[\Sigma] + k[S]\,. \tag{5.35}$$

In this manner, we can continue to alternately apply R_S and R_Σ, and get:

$$\begin{aligned}
R_S((2k^2-1)[\Sigma] + k[S]) &= (2k^2-1)[\Sigma] + (2k^3-2k)[S] \\
R_\Sigma((2k^2-1)[\Sigma] + (2k^3-2k)[S]) &= (4k^4-6k+1)[\Sigma] + (2k^3-2k)[S] \\
R_S((4k^4-6k+1)[\Sigma] + (2k^3-2k)[S]) &= (4k^4-6k+1)[\Sigma] + (4k^5-8k^3+3k)[S] \\
\vdots &= \vdots\,.
\end{aligned}$$

We observe that as long as $k \geq 2$, the polynomials above keep growing, thus implying that there is an infinite number of spheres with homology classes of the form $x = s_1[\Sigma] + s_2[S]$ with $x^2 = -1$. This cannot occur, since if it did, it would imply that there is an infinite number of Seiberg-Witten basic classes of the manifold M, which cannot happen if $b_2^+(M) > 1$. $\square$

Lemma 5.63 immediately implies the following Corollary:

Corollary 5.65. *With the intersection numbers $I_j = [\Sigma_{-1}] \cdot [S_j]$, as in section 5.3.2, we must have $I_2 + I_3 + I_4 + \cdots + I_{n-1} \leq 1$.*

Proof. The spheres S_j with $2 \leq j \leq n-1$ intersect transversally with the neighboring spheres in the plumbing configuration C_n. Therefore, we can construct the sphere S_2^{n-1}, which is the union of the spheres $S_j, 2 \leq j \leq n-1$, with all the transverse intersection points smoothed out. The sphere S_2^{n-1} has self-intersection (-2), since its homology class is:

$$[S_2^{n-1}] = [S_2] + [S_3] + [S_4] + \cdots + [S_{n-1}]\,. \tag{5.36}$$

Now we can apply Lemma 5.63 with $\Sigma = \Sigma_{-1}$ and $S = S_2^{n-1}$, and conclude that $[\Sigma_{-1}] \cdot [S_2^{n-1}]$ is at most 1, implying:

$$[\Sigma_{-1}] \cdot [S_2^{n-1}] = I_2 + I_3 + I_4 + \cdots + I_{n-1} \leq 1\,. \tag{5.37}$$

□

As a direct consequence of Corollary 5.65 and (5.28), we have the following:

Corollary 5.66. *If $c_1(X, \omega) \cdot \gamma > 0$, with $\gamma = nD + e^2$ as defined in section 5.3.2, then there is only one j, $1 \leq j \leq n-1$, for which $I_j = 1$ and $I_k = 0$ if $j \neq k$.*

Next, we will use toric and almost-toric fibrations, introduced in section 5.2.1, to show that for those cases where $c_1(X, \omega) \cdot \gamma > 0$, we have $\omega \cdot \gamma > 0$.

Proposition 5.67. *If there is only one j, $1 \leq j \leq n-1$, for which $I_j = 1$ and $I_k = 0$ if $j \neq k$, then $\omega \cdot \gamma > 0$.*

Proof. If there is only one j, $1 \leq j \leq n-1$, for which $I_j = 1$ and $I_k = 0$ if $j \neq k$, then by definition, the sphere Σ_{-1} only intersects the sphere S_j of the C_n configuration once at a point a_j. We can present part of Σ_{-1} as it intersects S_j, by a *visible surface* (see Definition 5.29), with the curve μ_j^1 and a compatible covector $u_j = \begin{bmatrix} -1 \\ n+1 \end{bmatrix}$, for all $1 \leq j \leq n-1$, (see Figure 68). We have $[\Sigma_{-1}] \cdot [S_j] = 1$, since $|u_j \times v_j| = 1$ for

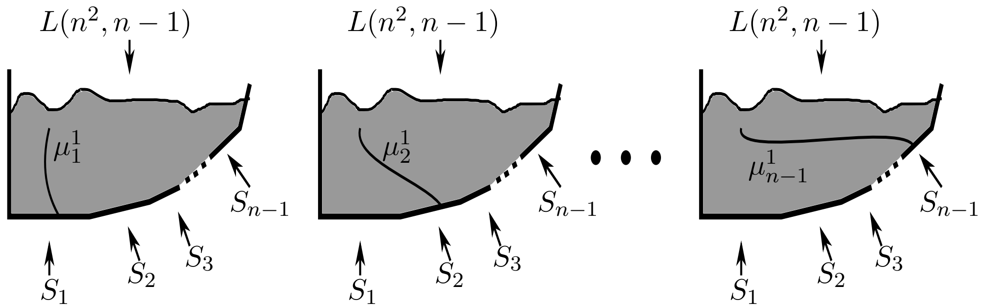


FIGURE 68. Visible surfaces represented by curves μ_j^1 in a toric model for C_n

all j, where the $v_j = \begin{bmatrix} 1-j \\ (j-1)n+j \end{bmatrix}$ are the *collapsing covectors* corresponding to the part of the 1-stratum that represents the spheres S_j, thus satisfying item (2) in the definition of visible surfaces (Definition 5.29).

After we perform the rational blow-down, as we do in the beginning of *Step 2*, we obtain the almost-toric base, as seen in Figure 69 (also see Figure 66). We recall here that the class $\gamma = nD + e^2$, where D is the "remains" of Σ_{-1} in X: $D = (X'/N(C_n)) \cap \Sigma_{-1}$. Since Σ_{-1} is a symplectic sphere, we have $\omega \cdot nD > 0$. In order to show that $\omega \cdot \gamma > 0$, we need to show that ω is positive on the 2-cell, e^2, which "closes up" nD i.e. $\partial e^2 = \partial nD$. We will do this by exhibiting the disk $e^2 \in B_n$ as a visible surface in the almost-toric fibration of B_n.

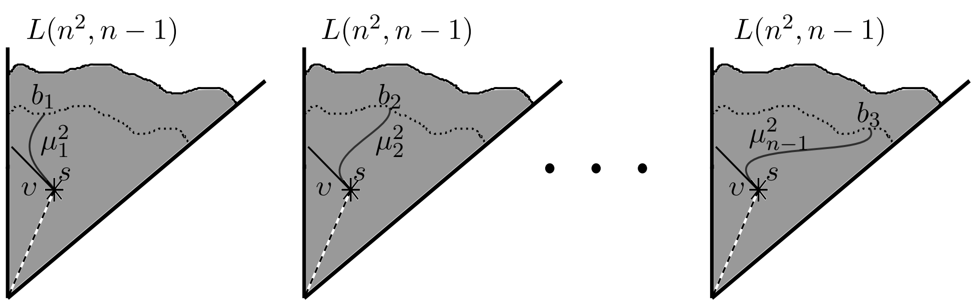


FIGURE 69. Visible surfaces represented by curves μ_j^2 in almost-toric model for B_n

In order to represent $e^2 \subset \gamma$ as a visible surface in the almost-toric base in Figure 69, we need to choose a curve μ_j^2, such that it extends the curve μ_j^1 and whose collection of compatible classes in $H_2(F_b, \mathbb{Z})$, for all $b \in \mu_j^2$, forms a disk. We can arrange the visible surface represented by μ_j^2 to be symplectic, because of Proposition 5.32. Also, we can arrange e^2 such that it hits the Lagrangian core $\mathcal{L}_{n,1}$, represented by the straight line v, thus the curve μ_j^2 hits v and then the node s.

On one hand, the compatible class of μ_j^2 must be the same as the vanishing class of the node s, in order for μ_j^2 to represent a visible surface (and be a disk). On the other hand, the curve μ_j^2 is a continuation of the curve μ_j^1. However, the curve μ_j^1 represents the visible surface for $D \in C_n$. At the point b_j in the toric fibration, which lies on the curve representing the boundary $\partial C_n = L(n^2, n-1)$, the compatible covector is $u_j = \begin{bmatrix} -1 \\ n+1 \end{bmatrix}$, corresponding to the class $\partial D \in L(n^2, n-1)$. When we begin the curve μ_j^2, at the point b_j, the compatible class should correspond to $n\partial D \in L(n^2, n-1)$, making the compatible covector $n \begin{bmatrix} -1 \\ n+1 \end{bmatrix} = \begin{bmatrix} -n \\ n^2+n \end{bmatrix}$. In $\partial C_n = \partial B_n = L(n^2, n-1)$, we have the compatible class with the covector $\begin{bmatrix} -n \\ n^2+n \end{bmatrix}$ homologous to the compatible class with the covector $\begin{bmatrix} -1 \\ n \end{bmatrix}$. As a result, the curve μ_j^2 will have a compatible covector $u'_j = \begin{bmatrix} -1 \\ n \end{bmatrix}$ for all $1 \leq j \leq n-1$, exactly the same as the vanishing covector of the node s. Consequently, the curves μ_j^2 do indeed represent visible surfaces, the 2-cells e^2 in the construction of γ.

As a result, we have explicitly exhibited that the class $\gamma = nD + e^2$ is such that $\omega \cdot \gamma > 0$, by representing D and e^2 as visible surfaces in the almost-toric fibrations of C_n and B_n, with positive symplectic area. □

Corollary 5.68. *Let γ be as above. If $c_1(X, \omega) \cdot \gamma > 0$, then $\omega \cdot \gamma > 0$.*

Proof. This is a direct consequence of Corollary 5.66 and Proposition 5.67. □

As a result of Corollary 5.68, we have proved that embeddings of $B_n \hookrightarrow X$ of type $\mathcal{A}_1 \subset \mathcal{A}$ cannot occur, where $\mathcal{A}_1$ is the set of $(n-1)$-tuples $\langle \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1} \rangle$, such that $c_1(X) \cdot \gamma > 0$ in terms of the intersection numbers $I_j = \alpha_j$.

5.3.4. *Step 4.* In this final step, we will show that symplectic embeddings of $B_n \hookrightarrow X$ of type $(\mathcal{A} - \mathcal{A}_1)$ and type $\mathcal{E}_k$, $k \geq c_1^2(X, \omega) + 2$ cannot occur. These sets of $(n-1)$-tuples precisely correspond with $c_1(X, \omega) \cdot \gamma \leq 0$, i.e. the cases where:

$$n - I_{n-1} - 2I_{n-2} - 3I_{n-3} - \cdots - (n-2)I_2 - (n-1)I_1 \leq 0. \tag{5.38}$$

Lemma 5.69. *Let $I_1 = [\Sigma_{-1}] \cdot [S_1]$ be as above, then $I_1 \leq n$.*

Proof. First, we apply the generalized adjunction formula (Theorem 5.46) to the sphere Σ_{-1}, which gives us that $c_1(X', \omega') + 2[\Sigma_{-1}]$ is a SW basic class of X'. Second, we apply Theorem 5.46 to the sphere S_1, since $[S_1]^2 = -n - 2$ for a SW basic class L we have:

$$|L \cdot [S_1]| \leq n \tag{5.39}$$

if we let $L = c_1(X', \omega') + 2[\Sigma_{-1}]$, then we have:

$$|(c_1(X', \omega') + 2[\Sigma_{-1}]) \cdot [S_1]| = |c_1(X', \omega') \cdot [S_1] + 2[\Sigma_{-1}] \cdot [S_1]| \leq n\,. \tag{5.40}$$

Since S_1 is a symplectic sphere, we have $c_1(X', \omega') \cdot [S_1] = -n$, therefore, we must have: $I_1 = [\Sigma_{-1}] \cdot [S_1] \leq n$. □

Corollary 5.70. *Let $I_i = [\Sigma_{-1}] \cdot [S_i]$, $1 \leq i \leq n-1$, be as above, then $I_1 + I_2 + I_3 + \cdots + I_{n-1} \leq n$.*

Proof. The spheres S_i intersect each other transversally in the C_n configuration. We can (as done in Corollary 5.65) construct the sphere S_1^{n-1}, which is the union of the spheres S_i, $1 \leq i \leq n-1$, with all the intersection points smoothed out (this can be done symplectically). The self-intersection number of the sphere S_1^{n-1} is $(-n-2)$, since its homology class is:

$$[S_1^{n-1}] = [S_1] + [S_2] + [S_3] + \cdots + [S_{n-1}]. \tag{5.41}$$

Consequently, by applying Lemma 5.69, since $I_1 \leq n$ then so is $I_1+I_2+I_3+\cdots+I_{n-1} \leq n$. □

In light of Corollaries 5.65, 5.70 and Lemma 5.69, the intersection patterns of Σ_{-1} with the spheres of the C_n configuration, giving us $c_1(X,\omega)\cdot\gamma \leq 0$, which we still have to rule out are:

(1) $2 \leq I_1 \leq n$ and $I_j = 0$ for all $2 \leq j \leq n-1$
(2) $1 \leq I_1 \leq n-1$ and $I_j = 1$ for one $2 \leq j \leq n-1$.

Lemma 5.71. *The following intersection configurations:*

(a) *$I_1 = n$ and $I_j = 0$ for all $2 \leq j \leq n-1$*
(b) *$I_1 = 1$ and $I_{n-1} = 1$ ($I_j = 0$ for all $2 \leq j \leq n-2$)*

will force the 4-manifold (X,ω) to have basic classes in addition to $\pm K = \mp c_1(X,\omega)$, thus contradicting the hypothesis in Theorem 5.6.

Proof. We begin with looking at the piece of the relative homology long exact sequence for the pair $(C_n, \partial C_n)$:

$$0 \to H_2(C_n;\mathbb{Z}) \xrightarrow{i} H_2(C_n,\partial C_n;\mathbb{Z}) \xrightarrow{\partial} H_1(\partial C_n;\mathbb{Z}) \to 0 \tag{5.42}$$

Let $\delta \in H_2(C_n,\partial C_n;\mathbb{Z})$ be a relative class that is a union of the disks $\Sigma_{-1}\cap N(C_n)$, where $N(C_n)$ is a neighborhood of the spheres of the C_n. For all $1 \leq j \leq n-1$, we have:

$$\delta\cdot[S_j] = \Sigma_{-1}\cdot[S_j] = I_j\,. \tag{5.43}$$

In case (a), for the class $-n\delta$ we have:

$$\begin{aligned} -n\delta\cdot[S_1] &= -n^2 \\ -n\delta\cdot[S_2] &= 0 \\ \vdots &= \vdots \\ -n\delta\cdot[S_{n-1}] &= 0\,. \end{aligned}$$

The relative class $-n\delta$ can be supported in the interior by the following homology class:

$$-n\delta = (n-1)[S_1] + (n-2)[S_2] + \cdots + [S_{n-1}] \tag{5.44}$$

since,

$$\begin{aligned} ((n-1)[S_1] + (n-2)[S_2] + \cdots + [S_{n-1}])\cdot[S_1] &= -n^2 \\ ((n-1)[S_1] + (n-2)[S_2] + \cdots + [S_{n-1}])\cdot[S_2] &= 0 \\ \vdots &= \vdots \\ ((n-1)[S_1] + (n-2)[S_2] + \cdots + [S_{n-1}])\cdot[S_{n-1}] &= 0\,. \end{aligned}$$

In case (b), for the class $-n\delta$ we have:

$$\begin{aligned} -n\delta\cdot[S_1] &= -n \\ -n\delta\cdot[S_2] &= 0 \\ \vdots &= \vdots \\ -n\delta\cdot[S_{n-2}] &= 0 \\ -n\delta\cdot[S_{n-1}] &= -n\,. \end{aligned}$$

In this case, the relative class $-n\delta$ can be supported in the interior by the following homology class:

$$-n\delta = [S_1] + 2[S_2] + \cdots + (n-1)[S_{n-1}] \tag{5.45}$$

since,

$$\begin{aligned}
([S_1] + 2[S_2] + \cdots + (n-2)[S_{n-2}] + (n-1)[S_{n-1}]) \cdot [S_1] &= -n \\
([S_1] + 2[S_2] + \cdots + (n-2)[S_{n-2}] + (n-1)[S_{n-1}]) \cdot [S_2] &= 0 \\
\vdots &= \vdots \\
([S_1] + 2[S_2] + \cdots + (n-2)[S_{n-2}] + (n-1)[S_{n-1}]) \cdot [S_{n-2}] &= 0 \\
([S_1] + 2[S_2] + \cdots + (n-2)[S_{n-2}] + (n-1)[S_{n-1}]) \cdot [S_{n-1}] &= -n\,.
\end{aligned}$$

In both cases (a) and (b) we have the relative class $-n\delta \in im(i) = ker(\partial)$, implying that $\partial(\delta) \in H_1(\partial C_n;\mathbb{Z}) \cong H_1(L(n^2, n-1);\mathbb{Z}) \cong \mathbb{Z}/n^2\mathbb{Z}$ is an element of order n. According to Theorem 5.43 (also see (5.1) and [Pa1]), this implies that the basic classes $\pm(c_1(X', \omega') + 2[\Sigma_{-1}])$ of (X', ω') extend to basic classes of the rational blow-down (X, ω). Note, the classes $\pm c_1(X', \omega')$ must extend to the basic classes of (X, ω) since they are the $\pm$ the canonical class. Moreover, $c_1(X', \omega') + 2[\Sigma_{-1}]$ and $c_1(X', \omega')$ must extend to different basic classes on (X, ω), otherwise we would have $[\Sigma_{-1}] = 0$. Therefore, the 4-manifold X will have at least four basic classes, which is a contradiction. ☐

Lemma 5.72. *The following intersection configurations:*

i) $I_1 = 1$ *and* $I_j = 1$ *for one* j *such that* $2 \leq j \leq n-2$

ii) $I_1 = k$ *with* $2 \leq k \leq n-1$ *and* $I_j = 1$ *for one* j *such that* $2 \leq j \leq n-1$

cannot occur in (X', ω'), *since it is a symplectic* 4*-manifold with* $b_2^+(X') > 1$.

Proof. Assume $I_1 = k$ for $1 \leq k \leq n-1$ and let $K = c_1(X', \omega')$ be the (negative of the) canonical class of (X', ω'). Since S_1 is a symplectic sphere with self-intersection $(-n-2)$, we have that $K \cdot [S_1] = -n$. Let L be any SW basic class of X', then according to the generalized adjunction formula for immersed spheres, Theorem 5.46, we have that

$$|L \cdot [S_1]| \leq n \tag{5.46}$$

or $SW_{X'}(L + 2[S_1]) = SW_{X'}(L)$ if $L \cdot S_1 \geq 0$, $(SW_{X'}(L - 2[S_1]) = SW_{X'}(L)$ if $L \cdot [S_1] \leq 0)$. We will produce a specific SW basic class L which will fail to satisfy (5.46) and for which $L \pm 2S_1$ cannot be a SW basic class since the symplectic 4-manifold X' is of simple type.

First, we observe, that by smoothing out the transverse intersections of the spheres in the C_n configuration and the sphere Σ_{-1}, we have the following spheres in X', each with self-intersection (-1):

$$\begin{aligned}
&\Sigma_{-1} \\
&\Sigma_{-1} + S_j \\
&\Sigma_{-1} + S_j + S_{j-1} \\
&\quad\vdots \\
&\Sigma_{-1} + S_j + S_{j-1} + \cdots + S_2 \\
&\Sigma_{-1} + S_j + S_{j-1} + \cdots + S_2 + S_{j+1} \\
&\Sigma_{-1} + S_j + S_{j-1} + \cdots + S_2 + S_{j+1} + S_{j+2} \\
&\quad\vdots \\
&\Sigma_{-1} + S_j + S_{j-1} + \cdots + S_2 + S_{j+1} + S_{j+2} + \cdots + S_{n-1}.
\end{aligned} \tag{5.47}$$

Second, using these spheres, we can construct several SW basic classes using Theorem 5.46 as follows: We start off by letting $L = K$ and $x = \Sigma_{-1}$ as in Theorem 5.46, since $|K \cdot [\Sigma_{-1}]| \leq -1$ cannot happen, $K + 2[\Sigma_{-1}]$ must be a SW basic class. Note, that $K^2 = (K + 2[\Sigma_{-1}])^2$, as required for 4-manifolds of simple type. Next, we let $L = K + 2[\Sigma_{-1}]$ and $x = \Sigma_{-1} + S_j$, and after applying Theorem 5.46 again, we get that since $|(K + 2[\Sigma_{-1}]) \cdot ([\Sigma_{-1}] + [S_j])| \leq -1$ cannot happen, then $(K + 2[\Sigma_{-1}]) + 2([\Sigma_{-1}] + [S_j])$ is a SW basic class. Proceeding in this manner, with all the spheres of (5.47), we get that K' is a SW basic class of X', where K' is:

(5.48)
$$K' = K+2(n-1)[\Sigma_{-1}]+2(n-2)[S_j]+\cdots+2(n-j)[S_2]+2(n-(j+1))[S_{j+1}]+\cdots+2[S_{n-1}].$$

Next, we again apply Theorem 5.46 with $L = K'$ and $x = S_1$, and as in (5.46), we get:

$$\begin{aligned} |K' \cdot [S_1]| &= |K \cdot [S_1] + 2(n-1)[\Sigma -1] \cdot [S_1] + 2(n-j)[S_2] \cdot [S_1]| \\ &= |(2k+1)n - 2k - 2j| \leq n. \end{aligned} \tag{5.49}$$

If $k = 1$, then (5.49) becomes:

$$|2n - 2 - 2j| \leq n\,, \tag{5.50}$$

which for $n \geq 4$ and $2 \leq j \leq n-2$ cannot occur. Therefore, $K' + 2[S_1]$ is forced to be a SW basic class, however, this is impossible since X' is of simple type and $(K' + 2[S_1])^2 \neq (K')^2$. Consequently, the configurations with intersection numbers $I_1 = 1$ and $I_j = 1$ for one j for which $2 \leq j \leq n-2$ cannot occur.

If $2 \leq k \leq n-1$, then the inequality (5.49) cannot hold if $2 \leq j \leq n-1$. Therefore, again $K' + 2[S_1]$ must be a SW basic class, but this cannot happen either since X' is of simple type. Consequently, the configurations with the intersection numbers

$I_1 = k$ with $2 \leq k \leq n-1$ and $I_j = 1$ for one j for which $2 \leq j \leq n-1$ cannot occur. $\square$

The results in section 5.3.2 as well as Lemmas 5.71 and 5.72 imply that if $n \geq c_1^2(X,\omega)+2$, then there cannot be symplectic embeddings of $B_n \hookrightarrow (X,\omega)$ of type $\mathcal{A}$. The only configurations which remain are those with $I_1 = k$ where $2 \leq k \leq n-1$ and $I_j = 0$ for j with $2 \leq j \leq n-1$, which correspond to symplectic embeddings of $B_n \hookrightarrow X$ of type $\mathcal{E}_k$ for $2 \leq k \leq n-1$. Next, we will show that symplectic embeddings $B_n \hookrightarrow (X,\omega)$ of type $\mathcal{E}_k$, $k \geq c_1^2(X,\omega)+2$, cannot occur.

Remark 5.73. The key difference between symplectic embeddings of $B_n \hookrightarrow X$ of type $\mathcal{A}$ and $\mathcal{E}_k$ is that in the embeddings of type $\mathcal{E}_k$, the sphere Σ_{-1} does not intersect any sphere with self-intersection (-2), which as seen in Lemma 5.72, creates quite a few Seiberg-Witten basic classes leading to contradictions because of adjunction formulas. Therefore, in order to prevent embeddings of type $\mathcal{E}_k$, $k \geq c_1^2(X,\omega)+2$, we need $c_1^2(X,\omega)$ to be low enough to guarantee the existence of several spheres with self-intersection (-1), in addition to Σ_{-1}.

The next Lemma will be instrumental in showing this last part of Theorem 5.6.

Lemma 5.74. *Let $S_r^d \subset (M,\omega)$ be an immersed symplectic sphere with self-intersection r and d double points, where (M,ω) is a symplectic 4-manifold with $c_1^2(M,\omega) \leq -1$ and $b_2^+(M) > 1$. Let $C_n^r \subset (M,\omega)$ be the linear plumbing of symplectic spheres $S_d^r, S_2, S_3, \ldots, S_{n-1}$, where the S_j are embedded symplectic spheres with $[S_j]^2 = -2$ for $2 \leq j \leq n-1$. Then there exists an embedded symplectic sphere $\hat{\Sigma}'_{-1} \subset (M,\omega)$ with $[\hat{\Sigma}'_{-1}]^2 = -1$, and $C_n'^r \subset (M,\omega)$, a linear plumbing configuration of symplectic spheres $S_r^d, S_2', S_3', \ldots, S_{n-1}'$ (each S_i' is a perturbation of S_i), such that if $\hat{\Sigma}'_{-1}$ intersects any spheres in the $C_n'^r$ configuration it must do so positively and transversally.*

Proof. The proof of this lemma mirrors the proof of Proposition 5.2 in section 5.3.1. As in the proof of Proposition 5.2, we start by putting an ω-compatible almost-complex structure J on the spheres of the C_n^r configuration. We can do so in the same manner as was done for the C_n configuration in Lemma 5.51. The only difference is that we apply Lemma 5.52 to the small Darboux neighborhoods of the double points of the immersed sphere S_r^d, as well as to the small Darboux neighborhoods of the intersections between adjacent spheres in the plumbing.

As before, since $c_1^2(M,\omega) \leq -1$, by Corollary 5.50 of the theorems of Taubes (Theorems 5.48 and 5.49), there must exist a J_ϵ-holomorphic sphere $\hat{\Sigma}_{-1}^\epsilon$ in (M,ω), with $[\hat{\Sigma}_{-1}^\epsilon]^2 = -1$ for a generic ω-compatible almost-complex structure J_ϵ. As in section 5.3.1, the spheres of the C_n^r configuration are J-holomorphic curves, and the sphere $\hat{\Sigma}_{-1}^\epsilon$ is a J_ϵ holomorphic curve. Therefore, we use Gromov Compactness (Theorem 5.57), and take a sequence of almost-complex structures $J_\epsilon \to J$ of which there exists a subsequence such that $\hat{\Sigma}_{-1}^\epsilon$ converges to a multicurve $\hat{u} = (\hat{u}^1, \ldots, \hat{u}^N)$. We can then apply Proposition 5.61, and conclude that there exists at least one i, such that $\hat{u}^i$ is an embedded J-holomorphic sphere, which we will label by $\hat{\Sigma}_{-1}$.

Again, as before, we apply Lemma 5.62, to the J-holomorphic curves S_r^d, S_2, S_3, $\ldots$, S_{n-1}, $\hat{\Sigma}_{-1}$, and perturb these into symplectic surfaces $\hat{S}_r^d$, S_2', S_3', $\ldots$, S_{n-1}', $\hat{\Sigma}'_{-1}$ which will intersect each other positively and transversally. The symplectic surface $\hat{S}_r^d$ has genus $g(\hat{S}_r^d) = d$, since it was obtained from the immersed sphere S_r^d by smoothing out the double points, see [LU]. However, we can replace $\hat{S}_r^d$ back with S_r^d, and consider the linear plumbing configuration of spheres $S_r^d, S_2', S_3', \ldots, S_{n-1}'$. We can still conclude that the sphere $\hat{\Sigma}'_{-1}$ (after a possible perturbation) intersects positively and transversally with that configuration, since S_r^d differs from $\hat{S}_r^d$ only in small neighborhoods around its double points. $\square$

Proposition 5.75. *Let $B_n \hookrightarrow (W,\omega)$, where (W,ω) is a symplectic 4-manifold with $b_2^+(W) > 1$, be an embedding of type $\mathcal{E}_k$, i.e. $I_1 = k$ and $I_j = 0$ for $2 \leq j \leq n-1$, for $k \geq c_1^2(W,\omega)+2$, then (W,ω) must have SW basic classes in addition to $\pm c_1(W,\omega)$.*

Proof. Assume $B_n \hookrightarrow (W,\omega)$ is an embedding of type $\mathcal{E}_k$. This implies that after symplectically rationally blowing up (W,ω), we obtain (W',ω') which contains a C_n configuration of symplectic spheres, and a symplectic sphere Σ_{-1} which intersects the sphere S_1 ($[S_1]^2 = -n-2$) k times positively and transversally.

We blow down the sphere Σ_{-1}, and obtain a manifold $(W^{(2)},\omega^{(2)})$, such that $c_1^2(W^{(2)},\omega^{(2)}) = c_1^2(W',\omega')+1$. The sphere $S_1 \subset W$ descends to an immersed sphere $S^{k-tuple}_{-n-2+k^2}$ which has self-intersection $(-n-2+k^2)$ and a k-tuple intersection point. Since the sphere S_1 was in fact pseudo-holomorphic, and the blow-down map is holomorphic, the immersed sphere $S^{k-tuple}_{-n-2+k^2}$ is pseudo-holomorphic as well. Therefore, $S^{k-tuple}_{-n-2+k^2}$ can be perturbed to a pseudo-holomorphic sphere with only double point intersections (see [Mc]), of which there will be $\frac{k(k-1)}{2}$ such double points. Consequently, the manifold $(W^{(2)},\omega^{(2)})$ will contain a linear configuration $C^{k(k-1)/2}_{-n-2+k^2}$ of spheres $S^{k(k-1)/2}_{-n-2+k^2}, S_2, S_3, \ldots, S_{n-1}$, where $S^{k(k-1)/2}_{-n-2+k^2}$ is an immersed symplectic sphere with self-intersection $r = -n-2+k^2$ and $d = \frac{k(k-1)}{2}$ double points.

Next, since $k \geq c_1^2(W,\omega)+2$, we have that $c_1^2(W^{(2)},\omega^{(2)}) \leq -1$, therefore, we can apply Lemma 5.74 and obtain an embedded symplectic sphere of self-intersection (-1): $\Sigma^{(2)}_{-1} \subset W^{(2)}$. This sphere $\Sigma^{(2)}_{-1}$ must intersect the configuration $C^{k(k-1)/2}_{-n-2+k^2}$, since if it did not, we could blow up $(W^{(2)},\omega^{(2)})$, obtain (W',ω') again, rationally blow down and get (W,ω), which would contain the sphere $\Sigma^{(2)}_{-1}$, a contradiction since $c_1^2(W,\omega) \geq 1$. By Lemma 5.74, $\Sigma^{(2)}_{-1}$ must then intersect the spheres of the $C^{k(k-1)/2}_{-n-2+k^2}$ configuration positively and transversally.

By Lemma 5.63, if $\Sigma^{(2)}_{-1}$ intersects with the spheres S_j, $2 \leq j \leq n-1$ and $[S_j]^2 = -2$, then we must have $[\Sigma^{(2)}_{-1}] \cdot [S_j] = 1$. However, if this is the case, then we would be able to blow down repeatedly $(n-2)$ times and end up with a manifold that has a sphere of self-intersection (-1) and $c_1^2 \geq 1$, which is a contradiction. Therefore, $\Sigma_{-1}^{(2)}$ must only intersect with the immersed sphere $S_{-n-2+k^2}^{k(k-1)/2}$.

Since $\Sigma_{-1}^{(2)}$ is a sphere of self-intersection (-1), then $c_1(W^{(2)}, \omega^{(2)}) + 2[\Sigma_{-1}^{(2)}]$ is a SW basic class of $W^{(2)}$, by Theorem 5.46. If we apply Theorem 5.46 to $x = S_{-n-2+k^2}^{k(k-1)/2}$, we obtain:

$$|(c_1(W^{(2)}, \omega^{(2)}) + 2[\Sigma_{-1}^{(2)}]) \cdot S_{-n-2+k^2}^{k(k-1)/2}| \leq n - k, \tag{5.51}$$

which implies that

$$[S_{-n-2+k^2}^{k(k-1)/2}] \cdot [\Sigma_{-1}^{(2)}] = j_2, \;\; 1 \leq j_2 \leq n - k. \tag{5.52}$$

If $j_2 = n - k$, then we could blow up $(W^{(2)}, \omega^{(2)})$ and obtain (W', ω'), which would now contain 2 spheres with self-intersection (-1): Σ_{-1} and $\Sigma_{-1}^{(2)}$, where:

$$\begin{aligned} [\Sigma_{-1}] \cdot [S_1] &= k \\ \left[\Sigma_{-1}^{(2)}\right] \cdot [S_1] &= n - k. \end{aligned}$$

Since $([\Sigma_{-1}] + [\Sigma_{-1}^{(2)}]) \cdot [S_1] = n$, as in Lemma 5.71, we can construct a relative class $\delta \in H_2(C_n, \partial C_n; \mathbb{Z})$ that is a union of the disks $(\Sigma_{-1} \cup \Sigma_{-1}^{(2)}) \cap N(C_n)$, such that the relative class $-n\delta$ can be supported in the interior by the following homology class:

$$-n\delta = (n-1)[S_1] + (n-2)[S_2] + \cdots + [S_{n-1}]. \tag{5.53}$$

As a result, just as in the proof of Lemma 5.71, the SW basic class $\pm(c_1(W', \omega') + 2[\Sigma_{-1}] + 2[\Sigma_{-1}^{(2)}])$ will extend to a SW basic class on (W, ω) after rationally blowing down, forcing (W, ω) to have basic classes in addition to $\pm c_1(W, \omega)$.

If $j_2 \neq n - k$, then we blow down the sphere $\Sigma_{-1}^{(2)}$ in $(W^{(2)}, \omega^{(2)})$, and obtain the manifold $(W^{(3)}, \omega^{(3)})$. The sphere $S_{-n-2+k^2}^{k(k-1)/2} \subset (W^{(2)}, \omega^{(2)})$ descends to the sphere $S_{-n-2+k^2+j_2^2}^{(k(k-1)+j_2(j_2-1))/2} \subset (W^{(3)}, \omega^{(3)})$, (after perturbing the j_2-tuple intersection, as done before). Next, since $k \geq c_1^2(W,\omega)+2$, we have that $c_1^2(W^{(3)},\omega^{(3)}) \leq -1$, therefore, we can apply Lemma 5.74 and obtain an embedded symplectic sphere of self-intersection (-1): $\Sigma_{-1}^{(3)} \subset W^{(3)}$. Again, we have that $c_1(W^{(3)},\omega^{(3)}) + 2[\Sigma_{-1}^{(3)}]$ is a SW basic class of $(W^{(3)},\omega^{(3)})$, thus by Theorem 5.46 we have that:

$$|(c_1(W^{(2)},\omega^{(2)}) + 2[\Sigma_{-1}^{(2)}]) \cdot S_{-n-2+k^2}^{k(k-1)/2}| \leq n-k, \tag{5.54}$$

which implies that

$$[S_{-n-2+k^2+j_2^2}^{(k(k-1)+j_2(j_2-1))/2}] \cdot [\Sigma_{-1}^{(3)}] = j_3, \;\; 1 \leq j_2 \leq n-k-j_2. \tag{5.55}$$

If $j_3 = n-k-j_2$, then we could blow up $(W^{(3)},\omega^{(3)})$ twice and obtain (W',ω'), which would now contain 3 spheres with self-intersection (-1): $\Sigma_{-1}, \Sigma_{-1}^{(2)}$ and $\Sigma_{-1}^{(3)}$, where:

$$\begin{aligned}
[\Sigma_{-1}] \cdot [S_1] &= k \\
\left[\Sigma_{-1}^{(2)}\right] \cdot [S_1] &= j_2 \\
\left[\Sigma_{-1}^{(3)}\right] \cdot [S_1] &= n-k-j_2.
\end{aligned}$$

Since $([\Sigma_{-1}] + [\Sigma_{-1}^{(2)}] + [\Sigma_{-1}^{(3)}]) \cdot [S_1] = n$, again as in Lemma 5.71, we can construct a relative class $\delta \in H_2(C_n,\partial C_n;\mathbb{Z})$ that is a union of the disks $(\Sigma_{-1} \cup \Sigma_{-1}^{(2)} \cup \Sigma_{-1}^{(3)}) \cap N(C_n)$, such that the relative class $-n\delta$ can be supported in the interior by the same class as before in (5.53). As a result, just as in the proof of Lemma 5.71, the SW basic class $\pm(c_1(W',\omega') + 2[\Sigma_{-1}] + 2[\Sigma_{-1}^{(2)}] + 2[\Sigma_{-1}^{(3)}])$ will extend to a SW basic class on (W,ω) after rationally blowing down, again forcing (W,ω) to have basic classes in addition to $\pm c_1(W,\omega)$.

If $j_3 \neq n-k-j_2$, we can repeat the same procedure again, which will again force (W,ω) to have basic classes in addition to $\pm c_1(W,\omega)$. We can continue this process until it terminates for some $\ell \leq n-k$, where we will have a j_ℓ so that

$j_2 + j_3 + j_4 + \cdots + j_\ell = n - k$. As a result, we will obtain the manifold $(W^{(\ell)}, \omega^{(\ell)})$, which will have a sphere $S^{(k(k-1)+j_2(j_2-1)+\cdots+j_{\ell-1}(j_{\ell-1}-1))/2}_{-n-2+k^2+j_2^2+\cdots+j_{\ell-1}^2}$ that interesects the sphere $\Sigma^{(\ell)}_{-1}$, $(n - k - j_2 - j_3 - \cdots - j_{\ell-1})$ times. We can then blow up $(W^{(\ell)}, \omega^{(\ell)})$ $(\ell - 1)$ times, and obtain the manifold (W', ω') which will have ℓ spheres of self-intersection (-1), such that:

$$
\begin{aligned}
[\Sigma_{-1}] \cdot [S_1] &= k \\
\left[\Sigma^{(2)}_{-1}\right] \cdot [S_1] &= j_2 \\
\left[\Sigma^{(3)}_{-1}\right] \cdot [S_1] &= j_3 \\
\vdots &= \vdots \\
\left[\Sigma^{(\ell-1)}_{-1}\right] \cdot [S_1] &= j_{\ell-1} \\
\left[\Sigma^{(\ell)}_{-1}\right] \cdot [S_1] &= n - k - j_2 - j_3 - \cdots - j_{\ell-1} = j_\ell.
\end{aligned}
$$

Again, in this case, we will have the SW basic class $\pm(c_1(W', \omega') + 2[\Sigma_{-1}] + 2[\Sigma^{(2)}_{-1}] + 2[\Sigma^{(3)}_{-1}] + \cdots + 2[\Sigma^{(\ell)}_{-1}])$ which will extend to a SW basic class on (W, ω) after rationally blowing down, again forcing (W, ω) to have basic classes in addition to $\pm c_1(W, \omega)$.

Notice, that we will have to do the greatest number of blow downs if $j_2 = j_3 = \cdots = j_\ell = 1$, in which case, $\ell = n - k$. Therefore, we require $k \geq c_1^2(W, \omega) + 2$, in order for all the manifolds $(W^{(i)}, \omega^{(i)})$ with $1 \leq i \leq \ell$ to have $c_1^2(W^{(i)}, \omega^{(i)}) \leq -1$, so that we can apply Lemma 5.74 repeatedly. □

From Proposition 5.75, we can see that if $B_n \hookrightarrow (X, \omega)$ is of type $\mathcal{E}_k$, $k \geq c_1(X, \omega) + 2$, then (X, ω) must have SW basic classes in addition $\pm c_1(X, \omega)$, which is a contradiction.

5.3.5. *Symplectic embeddings of type $\mathcal{E}_2$.* In this section we will show how to construct symplectic 4-manifolds (X, ω), such that the symplectic embeddings $B_n \hookrightarrow (X, \omega)$ are of type $\mathcal{E}_2$, for n odd. In these constructions (X, ω) will have $b_2^+(X) > 1$,

$n \geq c_1^2(X,\omega)+2$ and $\mathcal{B}as_X\,\{\pm(c_1(X,\omega)\}$. It is not clear however, whether such a construction actually yields a surface of general type or just a symplectic 4-manifold with said properties. First, we introduce the Fintushel and Stern knot surgery construction for 4-manifolds [FS3, FS4].

Definition 5.76. Let $T \subset X$ be a homologically non-trivial torus, with self-intersection 0, in a 4-manifold X with $b_2^+(X) > 1$. Let $T \times D^2$ be a tubular neighborhood of T in X. Also, let $K \subset S^3$ be a knot, and $N(K)$ be its tubular neighborhood. Then,

$$X_K = (X \backslash (T \times D^2)) \cup (S^1 \times (S^3 \backslash N(K))) \tag{5.56}$$

is defined to be the *knot surgery manifold.*

Note, the two pieces are attached in such a manner that the homology class $[* \times \partial D^2]$ is identified with $[* \times \lambda]$, where λ is the longitude of the knot K. In other words, X_K is obtained from X by removing a neighborhood of the torus T and replacing it with $(S^1 \times (S^3 \backslash N(K)))$. The manifold X_K is homotopy equivalent to X (assuming X is simply-connected).

In [FS3], Fintushel and Stern proved that the Seiberg-Witten invariants of X_K are determined by the Seiberg-Witten invariants of X and the Alexander polynomial of the knot K, as long as T has a cusp neighborhood. For the statement of this result, it is convenient to arrange all of the Seiberg-Witten basic classes into a Laurent polynomial as follows:

Definition 5.77. Let $\mathcal{B}as_X = \{\pm\beta_1, \ldots, \pm\beta_m\}$ and $t_{\beta_i} = exp(\beta_i)$ be variables satisfying $t_{\beta_i+\beta_j} = t_{\beta_i}t_{\beta_j}$, then

$$\mathcal{SW}_X = b_0 + \sum_{i=1}^{m} b_i(t_{\beta_i} + (-1)^{(\chi(X)+\sigma(X))/4} t_{\beta_i}^{-1}) \tag{5.57}$$

where $b_0 = SW_X(0)$ and $b_i = SW_X(\beta_i)$.

Example 5.78. Let $X = E(m)$ be the elliptic surface (see example 5.39), and $t = exp(T)$, where T is Poincaire dual of the fiber class, then:

$$\mathcal{SW}_{E(m)} = (t - t^{-1})^{m-2}\,. \tag{5.58}$$

Theorem 5.79. *Let $T \subset X$ be as above in Definition 5.76. Assume that T lies in a cusp neighborhood in X, then:*

$$\mathcal{SW}_{X_K} = \mathcal{SW}_X \cdot \Delta_K(t) \tag{5.59}$$

where $\Delta_K(t)$ is the Alexander polynomial of the knot K.

Remark 5.80. If $\Delta_K(t)$ is not monic then X_K cannot admit a symplectic structure, since if X_K is symplectic then we must have $SW_{X_K}(\pm c_1(X_K, \omega)) = \pm 1$. However, if the knot K is fibered, then the *knot surgery manifold* X_K has a symplectic structure [FS3], since it can be constructed as a symplectic fiber sum [Go1].

We will exhibit symplectic 4-manifolds which have symplectic embeddings $B_n \hookrightarrow X$ of type $\mathcal{E}_2$, by obtaining them from the elliptic surfaces $E(m)$ by knot surgery, blow-ups, and rational blow-down, (these constructions appeared in [Ak]). We will utilize the following Lefschetz fibration of the elliptic surfaces $E(m)$:

Lemma 5.81. [Ak] *There exists an elliptic Lefschetz fibration on the surface $E(m)$ with a section, a singular fiber F of type I_{8m}, $(2m-1)$ singular fibers of type I_2 and two additional fishtail fibers.*

Recall, that a singular fiber of type I_l is a plumbing of l spheres of self-intersection (-2) in a circle, and a fishtail fiber is an immersed sphere with one positive double point and self-intersection 0 (for more on elliptic surfaces and their singular fibers, see [HKK, KM], also see Figure 70).

In [FS4], Fintushel and Stern investigated the consequences of performing the *knot surgery* construction in certain neighborhoods in an elliptic fibration:

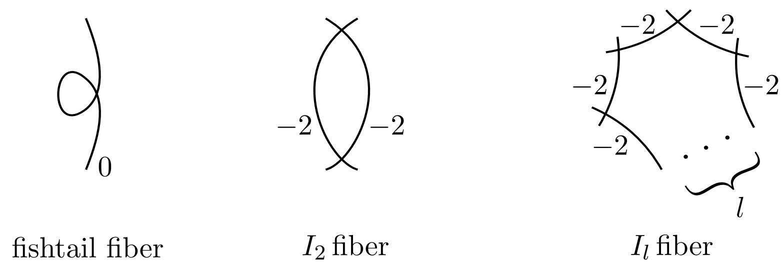


FIGURE 70. Fibers in an elliptic fibration

Definition 5.82. [FS4] A *double node neighborhood* D is a fibered neighborhood of an elliptic fibration which contains exactly two nodal fibers with the same monodromy.

One can perform knot surgery along a regular fiber in such a double node neighborhood, D, for example, in a neighborhood of the I_2 fiber (see Figer 70). The elliptic surface $E(m)$ will have a section R, which is a sphere with self-intersection $(-m)$. Fintushel and Stern observed that for a family of knots, the twists knots $T(r)$, if we perform knot surgery in the neighborhood of the I_2 singular fiber, then a disk in the section R gets replaced by a Seifert surface of the knot $T(r)$. As a result, the manifold $E(m)_{T(r)}$, will have a "pseudo-section" R_s, which we can think of as an immersed sphere with one double point (since $g(T(r)) = 1$), still having self-intersection $(-m)$ (see [FS4, Ak]). Note, we will use this construction only for the knot $T(1)$, which is the trefoil knot, since we are interested in our 4-manifolds retaining their symplectic structures.

Before we describe a family of symplectic 4-manifolds which will have a symplectic embedding of $B_n \hookrightarrow (X, \omega)$ of type $\mathcal{E}_2$, we first present some examples of a few of such manifolds. These constructions first appeared in [Ak], however, here we write them out in more detail.

Example 5.83. We consider the elliptic surface $E(3)$, which has $c_1^2(E(3)) = 0$, $b_2^+(E(3)) = 5$, $b_2^-(E(3)) = 29$, a section R with $[R]^2 = -3$, two fishtail fibers F_1 and F_2, five I_2 fibers and one I_{24} fiber in its fibration. We perform five knot surgeries,

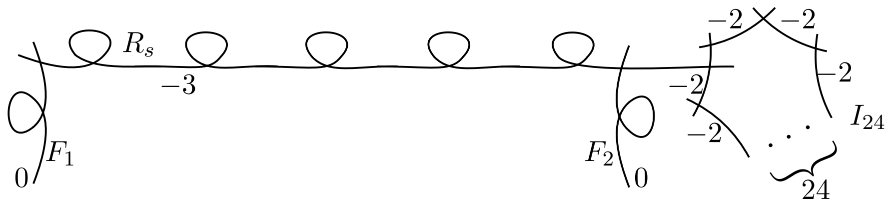


FIGURE 71. Pseudo-section R_s with fibers in $E(3)_{K_1,\dots,K_5}$

using the trefoil knot $T(1) = K_i$ each time in its five double node neighborhoods (the five I_2 fibers), and obtain the manifold $E(3)_{K_1,\dots K_5}$, which has a "pseudo-section" R_s, which is an immersed sphere with five double points having $[R_s]^2 = -3$. Additionally, this "pseudo-section" R_s has the fibers F_1, F_2 and I_{24} intersecting it, in the same way as they intersect R in $E(3)$, (see Figure 71).

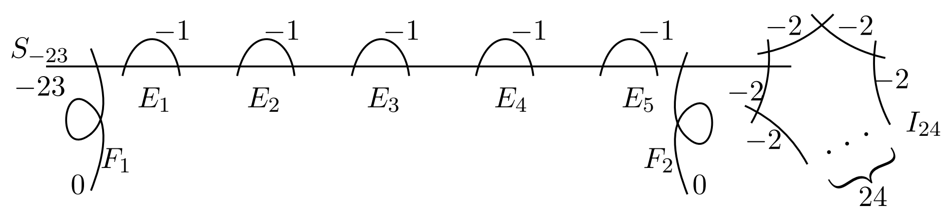


FIGURE 72. $E(3)_{K_1,\dots,K_5}\#5\overline{\mathbb{C}P^2}$

Next, we blow-up $E(3)_{K_1,\dots,K_5}$ at each of the five double points in the "pseudo-section" R_s, and obtain the manifold $E(3)_{K_1,\dots,K_5}\#5\overline{\mathbb{C}P^2}$. R_s becomes a sphere S_{-23} (self-intersection (-23)) in $E(3)_{K_1,\dots,K_5}\#5\overline{\mathbb{C}P^2}$. We have five exceptional spheres $E_1, \dots, E_5$, each of which intersects the sphere S_{-23} twice, (see Figure 72). As a result, in $E(3)_{K_1,\dots K_5}\#5\overline{\mathbb{C}P^2}$, we now have a C_{21} configuration of spheres, by taking the S_{-23} sphere together with 19 of the spheres in the I_{24} fiber. Finally, we can rationally blow down this C_{21} configuration and obtain a manifold $X_{(21)}$ with $b_2^+(X_{(21)}) = 5$, $b_2^-(X_{(21)}) = 14$ and $c_1^2(X_{(21)}) = 15$, such that:

$$B_{21} \hookrightarrow X_{(21)} = RBD(E(3)_{K_1,\dots,K_5}\#5\overline{\mathbb{C}P^2})$$

is a symplectic embedding of type $\mathcal{E}_2$. The embedding is symplectic, since we used a fibered knot, the trefoil, in the knot surgery, and the blow-up and rational blow-down operations are symplectic as well. It is of type $\mathcal{E}_2$ since we have a symplectic sphere (one of the exceptional spheres) intersecting the S_{-23} sphere of the C_{23} configuration twice. Moreover, the manifold $X_{(21)}$ has only one Seiberg-Witten basic class (up to sign), since only the top SW class of $E(3)_{K_1,\ldots,K_5}\#5\overline{\mathbb{C}P^2}$,

$$\pm(13)T + E_1 + \cdots + E_5$$

extends in the rational blow-down to $X_{(21)}$ (here, T is the Poincare dual of the fiber class, see [FS4, Ak, Pa1]).

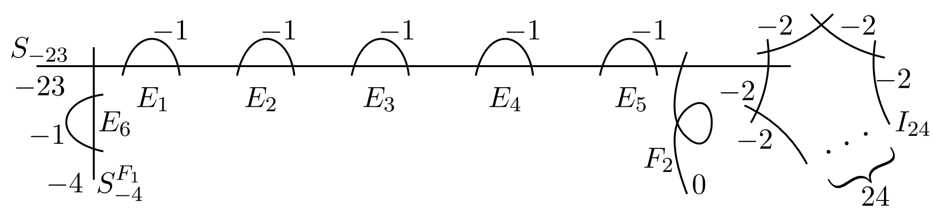


FIGURE 73. $E(3)_{K_1,\ldots,K_5}\#6\overline{\mathbb{C}P^2}$

Using variants of this method, we could also obtain embeddings of other rational homology balls B_n. We can take the manifold $E(3)_{K_1,\ldots,K_5}\#5\overline{\mathbb{C}P^2}$ and blow it up once more by blowing up one of the fishtail fibers (F_1) at the double point, and obtain $E(3)_{K_1,\ldots,K_5}\#6\overline{\mathbb{C}P^2}$, (see Figure 73). The fishtail fiber F_1 in $E(3)_{K_1,\ldots,K_5}\#5\overline{\mathbb{C}P^2}$ becomes a sphere with self-intersection (-4), $S^{F_1}_{-4}$ in $E(3)_{K_1,\ldots,K_5}\#6\overline{\mathbb{C}P^2}$ which intersects the sphere S_{-23} once. We can smooth out this intersection, and obtain a sphere S_{-25} (self-intersection (-25)), such that:

$$[S_{-25}] = [S^{F_1}_{-4}] + [S_{-23}]\,.$$

As a result, we have a C_{23} configuration of spheres in $E(3)_{K_1,\ldots,K_5}\#6\overline{\mathbb{C}P^2}$, by taking the sphere S_{-25} together with 21 spheres of the I_{24} fiber. After rationally blowing down the C_{23} configuration, we obtain a manifold $X_{(23)}$ with $b_2^+(X_{(23)}) = 5$,

$b_2^-(X_{(23)}) = 13$ and $c_1^2(X_{(23)}) = 16$, such that:

$$B_{23} \hookrightarrow X_{(23)} = RBD(E(3)_{K_1,\ldots,K_5}\#6\overline{\mathbb{C}P^2})$$

is a symplectic embedding of type $\mathcal{E}_2$, similar to the embedding of B_{21} before.

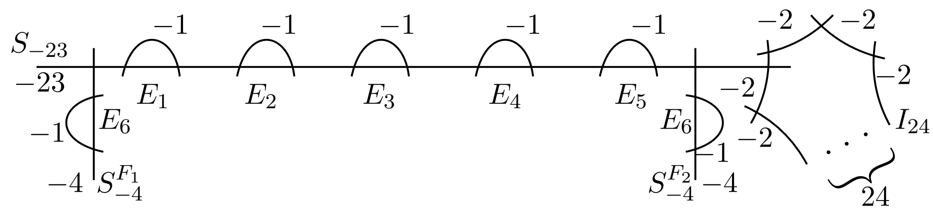


FIGURE 74. $E(3)_{K_1,\ldots,K_5}\#7\overline{\mathbb{C}P^2}$

Likewise, we can take the manifold $E(3)_{K_1,\ldots,K_5}\#6\overline{\mathbb{C}P^2}$ and blow it up once more by blowing up the other fishtail fiber (F_2) at the double point, and obtain the manifold $E(3)_{K_1,\ldots,K_5}\#7\overline{\mathbb{C}P^2}$, (see Figure 74). The fishtail fiber F_2 in $E(3)_{K_1,\ldots,K_5}\#6\overline{\mathbb{C}P^2}$ becomes a sphere with self-intersection (-4), $S_{-4}^{F_2}$ in $E(3)_{K_1,\ldots,K_5}\#7\overline{\mathbb{C}P^2}$ which intersects the sphere S_{-23} once. We can smooth out the intersections of S_{-23} with the spheres $S_{-4}^{F_1}$ and $S_{-4}^{F_2}$, and obtain a sphere S_{-27}, such that:

$$[S_{-27}] = [S_{-4}^{F_1}] + [S_{-23}] + [S_{-4}^{F_2}]\,.$$

As a result, we have a C_{25} configuration of spheres in $E(3)_{K_1,\ldots,K_5}\#7\overline{\mathbb{C}P^2}$, by taking the sphere S_{-27} together with 23 spheres of the I_{24} fiber. Again, after rationally blowing down the C_{25} configuration, we obtain a manifold $X_{(25)}$ with $b_2^+(X_{(25)}) = 5$, $b_2^-(X_{(25)}) = 12$ and $c_1^2(X_{(25)}) = 17$, such that:

$$B_{25} \hookrightarrow X_{(25)} = RBD(E(3)_{K_1,\ldots,K_5}\#7\overline{\mathbb{C}P^2}) \tag{5.60}$$

is a symplectic embedding of type $\mathcal{E}_2$, similar to the embeddings of B_{21} and B_{23} before.

Additionally, we can alter this type of construction by performing knot surgeries in one, two, three or four of the double node neighborhoods of the elliptic fibration of $E(3)$, instead of doing five knot surgeries as in the previous examples.

Next, we will describe the general construction of a family of such manifolds, similar to the examples above, (again, see [Ak]).

Proposition 5.84. *There exists a family of symplectic* 4*-manifolds* $\mathcal{X}$*, with each* $(X,\omega) \in \mathcal{X}$ *having* $b_2^+(X) > 1$, $\mathcal{B}as_X = \{\pm c_1(X,\omega)\}$ *and a symplectic embedding* $B_n \hookrightarrow (X,\omega)$ *of type* $\mathcal{E}_2$*, for* n *odd. Moreover, for all* $(X,\omega) \in \mathcal{X}$*, the embeddings of* $B_n \hookrightarrow (X,\omega)$ *are such that* $n < 3 + \frac{4}{3}c_1^2(X,\omega)$*.*

Proof. We proceed as in Example 5.83. First, we take the elliptic surface $E(m)$, $m > 2$, which has a section R, a sphere of self-intersection $(-m)$, and perform knot surgery in the double node neighborhoods of s of the I_2 fibers, obtaining the manifold $E(m)_{K_1,\ldots,K_s}$, where $1 \leq s \leq 2m-1$ and K_i are copies of the trefoil knot. We now obtain a "pseudo-section" R_s (see [FS4, Ak]) of $E(m)_{K_1,\ldots,K_s}$, which is an immersed sphere with self-intersection $(-m)$ and s double points.

As in Example 5.83, we can blow up s times, so that R_s becomes the embedded sphere S_{-m-4s} (self-intersection $(-m-4s)$) in $E(m)_{K_1,\ldots,K_s}\#s\overline{\mathbb{C}P^2}$. Additionally, in $E(m)_{K_1,\ldots,K_s}\#s\overline{\mathbb{C}P^2}$ we will have s exceptional spheres $E_1,\ldots,E_s$, with $[E_i]^2 = -1$, each of which intersects the sphere S_{-m-4s} twice. In the fibration of $E(m)$, we also have two additional fishtail fibers, F_1 and F_2 (Lemma 5.81), which intersect the "pseudo-section" R_s once. Therefore, we can blow up $E(m)_{K_1,\ldots,K_s}$ $(s+1)$ times (at the double points of R_s and the fishtail fiber F_1), and after smoothing out the transverse intersection, obtain a sphere $S_{-m-4s-2}$ in $E(m)_{K_1,\ldots,K_s}\#(s+1)\overline{\mathbb{C}P^2}$, such that $[S_{-m-4s-2}] = [S_{-4}^{F_1}] + [S_{-m-4s}]$. Likewise, we can blow up $E(m)_{K_1,\ldots,K_s}$ $(s+2)$ times (at the double points of R_s and the fishtail fibers F_1 and F_2), and after smoothing out the transverse intersections, obtain a sphere $S_{-m-4s-4}$ in $E(m)_{K_1,\ldots,K_s}\#(s+2)\overline{\mathbb{C}P^2}$, such that $[S_{-m-4s-4}] = [S_{-4}^{F_1}] + [S_{-m-4s}] + [S_{-4}^{F_2}]$. (The spheres $S_{-4}^{F_1}$ and $S_{-4}^{F_2}$ are the same as in Example 5.83.)

In these three cases, we obtain configurations of C_{m+4s-2}, C_{m+4s} and C_{m+4s+2} in

$$E(m)_{K_1,\ldots,K_s} \# s\overline{\mathbb{C}P^2}$$
$$E(m)_{K_1,\ldots,K_s} \# (s+1)\overline{\mathbb{C}P^2}$$
$$E(m)_{K_1,\ldots,K_s} \# (s+2)\overline{\mathbb{C}P^2},$$

respectively, by taking the spheres S_{-m-4s}, $S_{-m-4s-2}$ and $S_{-m-4s-4}$, also respectively, with the spheres of the I_{8m} fiber. Note, this can be done as long we have enough spheres of self-intersection (-2) in the I_{8m} fiber to complete the C_{m+4s-2}, C_{m+4s} and C_{m+4s+2} configurations, so we must have $(8m-1) \geq (m+4s)$, $(8m-1) \geq (m+4s-2)$ or $(8m-1) \geq (m+4s-4)$, respectively. We can then rationally blow down these configurations and obtain manifolds $X_{(m+4s-2)}$, $X_{(m+4s)}$ and $X_{(m+4s+2)}$, such that:

$$\begin{aligned} B_{m+4s-2} &\hookrightarrow X_{(m+4s-2)} \cong RBD(E(m)_{K_1,\ldots,K_s} \# s\overline{\mathbb{C}P^2}) \\ B_{m+4s} &\hookrightarrow X_{(m+4s)} \cong RBD(E(m)_{K_1,\ldots,K_s} \# (s+1)\overline{\mathbb{C}P^2}) \\ B_{m+4s+2} &\hookrightarrow X_{(m+4s+2)} \cong RBD(E(m)_{K_1,\ldots,K_s} \# (s+2)\overline{\mathbb{C}P^2}). \end{aligned}$$

In all of these cases, just like in Example 5.83, the embeddings of B_n will be symplectic (since we used the trefoil knot in the knot surgery construction) and will be of type $\mathcal{E}_2$ (due to the exceptional spheres E_i). Again, if m is odd, then only the top basic classes

$$\pm(m+2s-2)T + E_1 + E_2 + \cdots + E_r$$

of $E(m)_{K_1,\ldots,K_s} \# r\overline{\mathbb{C}P^2}$ extend to the rational blow-down, where $r \in \{s, s+1, s+2\}$, (this follows from results in [Pa1], also see [Ak]). As a result, the manifolds $X_{(m+4s-2)}$, $X_{(m+4s)}$ and $X_{(m+4s+2)}$ will each only have one SW basic class, up to sign.

It is clear from these embeddings of the rational homology balls B_n, that if we want higher values of n, we are going to have to take higher values of m, thus, we need to increase the b_2^+ betti number. In these constructions, the number n is mainly restricted by the number of spheres of self-intersection (-2) in the I_{8m} fiber which we use to construct the C_n configuration of spheres. Consequently, even if we use all of the $(2m-1)$ of the I_2 for our knot surgery construction along with both of the fishtail fibers F_1 and F_2, and get a sphere $S_{-m-4s-4}$, we may not be able not construct a C_n configuration of spheres with $n = m + 4s + 2$ if we have $(8m-1) < (n-1)$. For this reason, for each m, in order to get the highest possible value for n, we may have to use less than the $(2m-1)$ of the I_2 fibers in our knot surgery construction. Consequently, the highest n which will work for these constructions is when $n = 8m + 1$, where we use all the $(8m-1)$ available spheres of the I_{8m} fiber.

If $m = 4k + 1$, for $k \geq 1$, then we have:

$$B_{8m+1} \hookrightarrow X_{(8m+1)} \cong RBD(E(m)_{K_1,\ldots,K_{7k+2}} \# (7k+3)\overline{\mathbb{C}P^2}) \,,$$

where $b_2^+(X_{(8m+1)}) = 2m - 1$ and $c_1^2(X_{(8m+1)}) = 25k + 5$.

If $m = 4k + 3$, for $k \geq 1$, then we have:

$$B_{8m+1} \hookrightarrow X_{(8m+1)} \cong RBD(E(m)_{K_1,\ldots,K_{7k+6}} \# (7k+6)\overline{\mathbb{C}P^2}) \,,$$

where $b_2^+(X_{(8m+1)}) = 2m - 1$ and $c_1^2(X_{(8m+1)}) = 25k + 2$. The case $m = 3$ was done in Example 5.83, where $n = 25$ was the highest possible number, (see (5.60)).

As a result, we can see that as $(\chi_h, c_1^2) \to \infty$ then $n \to \infty$ as well. Moreover, in all these examples we have $n < 3 + \frac{4}{3}c_1^2$. If we take $m \geq 5$, we can refine this bound to $n < 3 + \frac{32}{25}c_1^2$. $\square$

It is important to note that it is not clear whether the examples in Proposition 5.84 yield surfaces of general type or just symplectic 4-manifolds. Additionally, as proposed in Conjecture 5.9, one could probably construct embeddings of type $\mathcal{E}_k$ for

$k \geq 3$ having the same properties as those of type $\mathcal{E}_2$ in Proposition 5.84. This might be done by defining the knot surgery construction in double node neighborhoods for fibered knots with higher genus than the trefoil knot.

6. Generalized rational homology balls $B_{n,m}$

Park [Pa1] has extended Fintushel and Stern's rational blow-down construction to a *generalized rational blow-down* construction. In this *generalized rational blow-down* construction, one takes a negative definite plumbing manifold $C_{n,m}$ (where $n \geq 2$, $m \geq 1$ and n and m are relatively prime), given by the diagram in Figure 75, where k is the length of the continued fraction expansion $[a_1, a_2, a_3, \ldots, a_k]$ of $\frac{n^2}{nm-1}$. In Figure 75, the spheres S_i, $1 \leq i \leq k$, have self-intersection numbers $(-a_i)$, where $a_i \geq 2$. Note, if $m = 1$ we recover the C_n configuration.

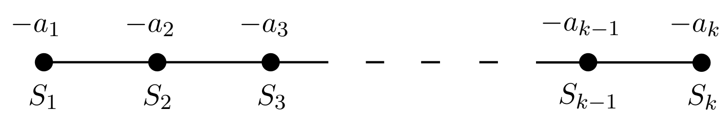


Figure 75. **Plumbing diagram of $C_{n,m}$, $n \geq 2$, $m \geq 1$**

It follows that the boundary of $C_{n,m}$ is the lens space $L(n^2, nm-1)$, thus again $\pi_1(\partial C_{n,m}) \cong H_1(\partial C_{n,m}; \mathbb{Z}) \cong \mathbb{Z}/n^2\mathbb{Z}$. These lens spaces also bound the *generalized rational homology balls* $B_{n,m}$, [CH]. Therefore, one can define the generalized rational blow-down in a similar manner as the standard one: remove the negative definite $C_{n,m}$ manifold and replace it with the rational homology ball $B_{n,m}$.

Symington in [Sy2, Sy3] showed that just like in the standard case, the generalized rational blow-down can be performed in the symplectic category, provided the spheres in the $C_{n,m}$ configuration are symplectic and intersect each other transversally. The almost-toric base (see section 5.2.1) for the rational homology balls $B_{n,m}$ can be presented as in Figure 76.

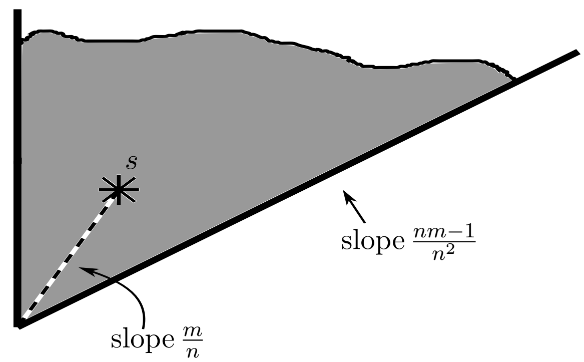


FIGURE 76. Almost-toric base for $B_{n,m}$

It would be interesting to investigate the possible obstructions to embedding these generalized rational homology balls $B_{n,m}$, both smoothly and symplectically. The greatest difficulty in generalizing the proof of Theorem 5.6 for embeddings of $B_{n,m}$ is perhaps that one cannot take advantage of the spheres of self-intersection (-2) which are present in the C_n configuration. As seen in Lemma 5.63, Corollary 5.65, Lemma 5.72 and Proposition 5.75, in the proof of Theorem 5.6, the fact that the spheres S_j, $2 \leq j \leq n-1$ in the C_n configuration have self-intersection (-2), plays a crucial role.

7. Appendices

Appendix A.

The following sequence of Kirby diagrams show the equivalence of the two different Kirby diagrams for the rational homology balls B_n, as seen in Figures 2 and 12. We start off with Figure 77, a Kirby diagram of B_n as in Figure 2, and illustrate the n positive twists in Figure 78. Next we add a cancelling 1/2-handle pair which includes a 0-framed two-handle, Figure 79. After this, we slide the $(n-1)$-framed handle off of the 0-framed handle, and obtain Figure 80, where the $(n-1)$-framed handle becomes a $(n-3)$-framed handle. We can continue to perform handleslides as seen in Figure 81 and Figure 82, until we have completely slid off the original two-handle from the original one-handle, obtaining Figure 83. Finally, we remove a cancelling 1/2-handle pair, and obtain Figure 84, with n negative twists, which corresponds to Kirby diagram Figure 85 (identical to Figure 12).

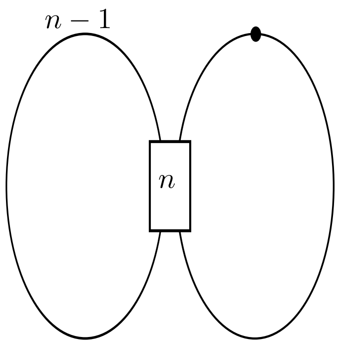


FIGURE 77.

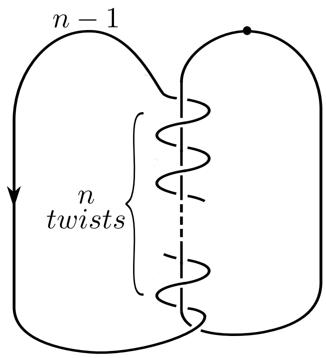


FIGURE 78.

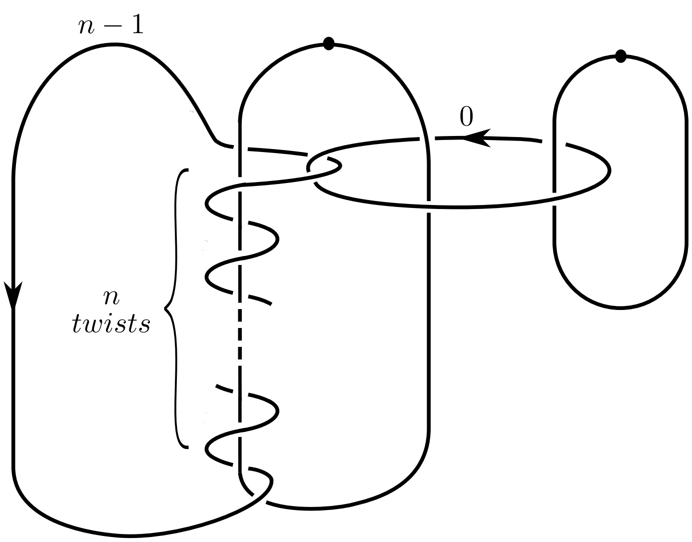


FIGURE 79.

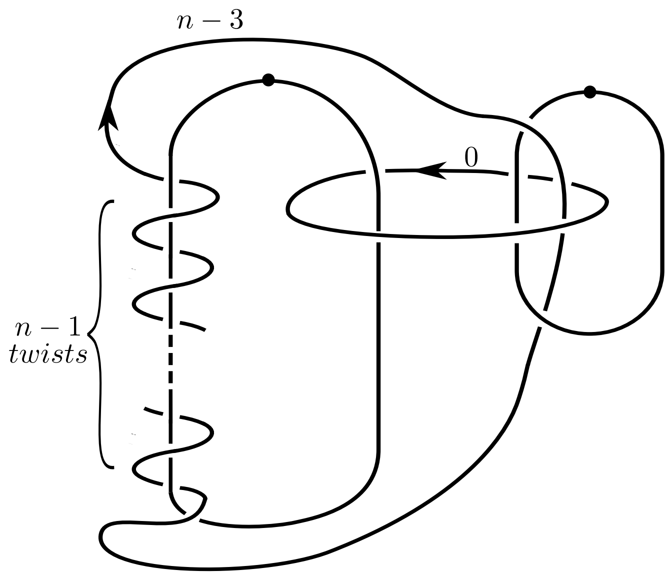


FIGURE 80.

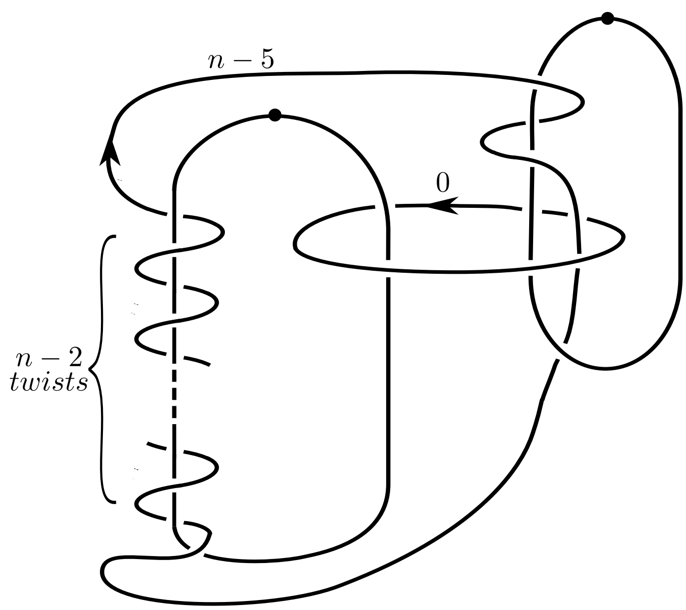


FIGURE 81.

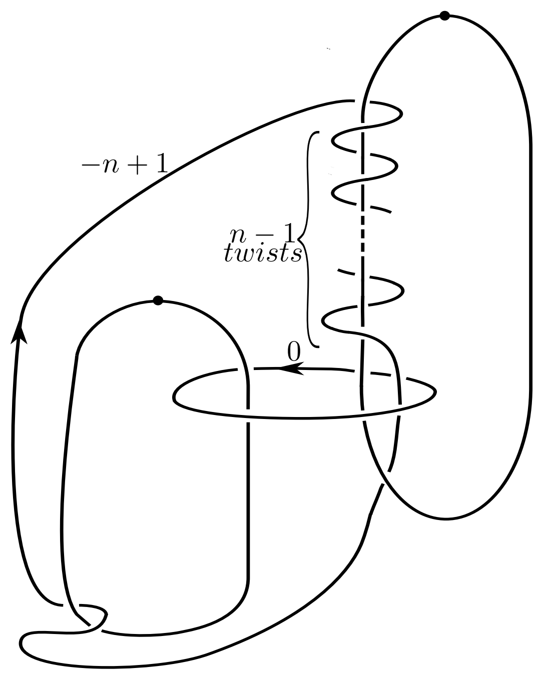


FIGURE 82.

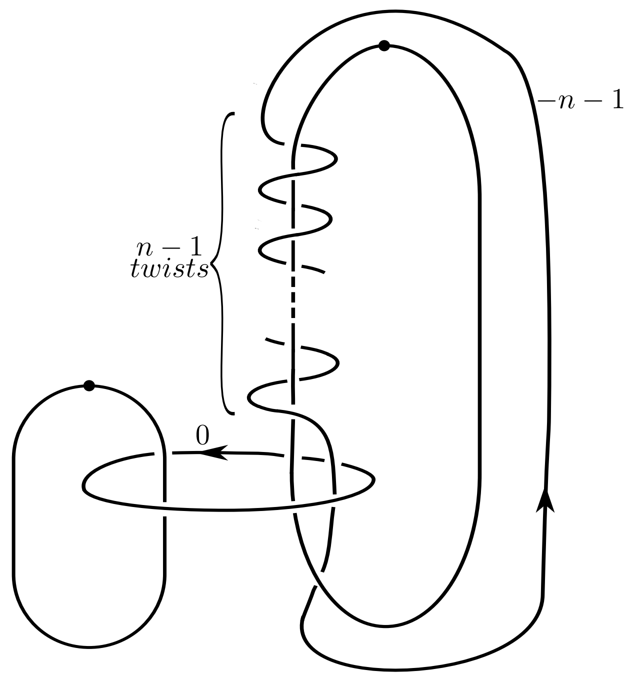


FIGURE 83.

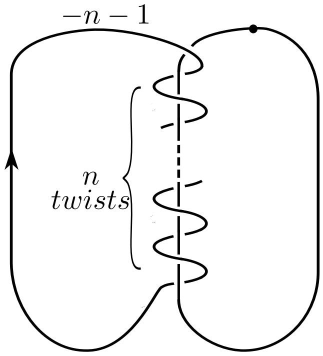


FIGURE 84.

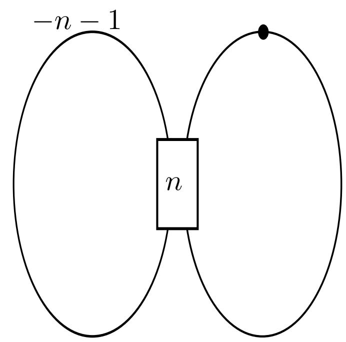


FIGURE 85.

References

[Ak] A. Akhmedov, *Construction of exotic smooth structures*, Topology and its Applications **154** (2007), no. 6, 1134-1140

[At] M. F. Atiyah. *Convexity and commuting Hamiltonians*, Bull. London Math. Soc., **14** (1982), no. 1, 1-15

[BM] M. Boucetta and P. Molino, *Géométrie globale des systèmes hamiltoniens complètement intégrables: fibrations lagrangiennes singulières et coordonnées action-angle à singularités.*, C. R. Acad. Sci. Paris Sér. I Math., **308** (1989), no. 13, 421-424

[Ca] A. Cannas da Silva, *Lectures on Symplectic Geometry*, Lecture Notes in Mathematics 1764, Springer-Verlag, 2001

[CH] A. Casson and J. Harer, *Some homology lens spaces which bound rational balls*, Pacific J. Math. **96** (1981), 23-36

[De] T. Delzant, *Hamiltoniens périodiques et images convexes de lapplication moment*, Bull. Soc. Math. France, **116** (1988), no. 3, 315-339

[El1] Y. Eliashberg, *Classification of overtwisted contact structures on* 3*-manifolds*, Invent. Math. **98**, (1989), 623-637

[El2] Y. Eliashberg, *Topological characterization of Stein manifolds of dimension* > 2, Int. J. of Math. **1** (1990), 29-46

[EM] Y. Eliashberg and N. Mishachev, *Introduction to the h-principle*, Graduate Studies in Mathematics 48, Amer. Math. Soc., Providence, 2002

[Et] J. B. Etnyre. *Legendrian and Transversal Knots*, in Handbook of Knot Theory, Elsevier B. V., Amsterdam, 2005, 105-185

[FS1] R. Fintushel and R. Stern, *Immersed spheres in* 4*-manifolds and the immersed Thom conjecture*, Turkish J. Math. **19** (1995), no. 2, 145-157

[FS2] R. Fintushel and R. Stern, *Rational blowdowns of smooth 4-manifolds*, J. Diff. Geom. **46** (1997) 181-235

[FS3] R. Fintushel and R. Stern, *Knots, links, and 4-manifolds*, Invent. Math. **134** (1998), 363400

[FS4] R. Fintushel and R. Stern, *Double node neighborhoods and families of simply connected* 4*-manifolds with* $b^+ = 1$, J. Amer. Math. Soc. **19** (2006), 171-180.

[Fr] M. Freedman, *The topology of four-dimensional manifolds*, J. Diff. Geom. **17** (1982), 357-453

[FQ] M. Freedman and F. Quinn, *Topology of* 4*-manifolds*, Princeton Mathematical Series 39, Princeton University Press, 1990

[FM] R. Friedman, J. W. Morgan. *On the diffeomorphism types of certain algebraic surfaces*, I. J. Diff. Geom., **27** (1988), no. 2, 297-369

[Ge] H. Geiges. *An Introduction to Contact Topology*, Cambridge University Press, 2008

[Gi] E Giroux, *Structures de contact en dimension trois et bifurcations des feuilletages de surfaces*, Invent. Math. **141** (2000), no. 3, 615-689.

[Go1] R. Gompf, *A new construction of symplectic manifolds*, Ann. of Math. **142** (1995), 527-595

[Go2] R. Gompf, *Handlebody construction of Stein surfaces*, Ann. of Math. **148** (1998), no. 2, 619-693

[GS] V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping* Invent. Math., **67** (1982), no. 3, 491-513

[Gr] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math., **82** (1985), 307-347

[GS] R. Gompf and A. Stipsicz, *An introduction to 4-manifolds and Kirby calculus*, Graduate Studies in Mathematics 20, (American Mathematical Society, Providence, RI, (1999))

[HKK] J. Harer, A. Kas and R. Kirby, *Handlebody decompositions of complex surfaces*, Memoirs AMS **62** (1986), no. 350

[Ho] K. Honda, *On the classification of tight contact structures I: Lens spaces, solid tori, and* $T^2 \times I$, Geom. Topol. **4** (2000), 309-368

[KM] R. Kirby and P. Melvin, *The* E_8 *manifolds, singular fibers and handlebody decomposition*, Algebr. Geom. Topol. **3** (2003), 577-568

[KSB] J. Kollár and N. I. Shepherd-Barron, *Threefolds and deformations of surface singularities*, Invent. Math. **91** (1988), no. 2, 299-338

[Ko] D. Kotschick, *The Seiberg-Witten invariants of symplectic four-manifolds (after C. H. Taubes)*, Seminaire Bourbaki, 1995/96, **241** (1997), no. 812, 195-220.

[KM] P. Kronheimer and T. Mrowka, *The genus of embedded surfaces in the projective plane*, Math. Res. Letters **1** (1994), no. 6, 797808

[LP] F. Laudenbach and V. Poénaru, *A note on* 4*-dimensional handlebodies*, Bull. Soc. Math. France **100** (1972), 337-344

[Li] T.-J. Li, *Existence of symplectic surfaces*, in Geometry and topology of manifolds, Fields Inst. Commun., 47, AMS Providence, RI, 2005, 203217

[LU] T.-J. Li, M. Usher, *Symplectic forms and surfaces of negative square*, J. Symplectic Geom. **4** (2006), no. 1, 71-91.

[Lis] P. Lisca, *On symplectic fillings of lens spaces*, Trans. Amer. Math. Soc. **360** (2008), 765-799

[Mc] D. McDuff, *Immersed spheres in symplecic 4-manifolds*, Ann. Inst. Fourier, Grenoble **42**, (1992), no. 1-2, 369-392

[McPo] D. McDuff & L. Polterovich, *Symplectic packings and algebraic geometry*, Invent. Math. **115** (1994), 405-429

[MS1] D. McDuff, & D.A. Salamon, *J-Holomorphic Curves and Quantum Cohomology*, AMS, University Lecture Series, Vol. 6, Providence, Rhode Island, 1994

[MS2] D. McDuff, & D.A. Salamon, *Introduction to Symplectic Topology*, second edition, Oxford University Press, 1998

[MS3] D. McDuff, & D.A. Salamon, *J-Holomorphic Curves and Symplectic Topology*, AMS Colloquium Publications, Vol. 52, 2004

[MW] M. Micallef and B. White, *The structure of branch points in minimal surfaces and in pseudo-holomorphic curves*, Ann. Math. **139** (1994), 35-85

[Mo] J. Morgan. *The Seiberg-Witten equations and applications to the topology of smooth four-manifolds*, Vol. 44 of Math. Notes. Princeton University Press, Princeton, NJ, 1996

[OzSt] B. Ozbagci and A. Stipsicz, *Surgery on contact 3-manifolds and Stein surfaces*, Bolyai Soc. Math. Stud., 13, Springer-Verlag, Berlin, 2004

[OzSz] P. Ozsváth and Z. Szabó, *The symplectic Thom conjecture*, Ann. math. **151** (2000), no. 2, 93-124

[Pa1] J. Park. *Seiberg-Witten invariants of generalised rational blow-downs.* Bull. Austral. Math. Soc., **56** (1997), no. 3, 363-384

[Pa2] J. Park: *Simply connected symplectic 4-manifolds with* $b_2^+ = 1$ *and* $c_1^2 = 2$, Invent. Math. **159** (2005), 657-667

[Po] L. Pontrjagin, *A classification of mappings of the three-dimensional complex into the two dimensional sphere*, Matematicheskii Sbornik **9** (1941), no. 2, 331-363

[SS] A. Stipsicz and Z. Szabó, *An exotic smooth structure on* $\mathbb{C}P^2\#6\overline{\mathbb{C}P^2}$, Geom. Topol. **9** (2005), 813-832

[Sy1] M. Symington, *Symplectic rational blowdowns*, J. Diff. Geom. **50** (1998), 505-518

[Sy2] M. Symington, *Generalized symplectic rational blowdowns*, Algebr. Geom. Topol. **1** (2001), 503-518

[Sy3] M. Symington, *Four dimensions from two in symplectic topology*, Topology and Geometry of manifolds (Athens, GA, 2001) Proc. Sympos. Pure Math. 71, Amer. Math. Soc., Providence, RI, 2003, 153-208

[Ta1] C. H. Taubes, *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Letters **1** (1994), 809-822

[Ta2] C. H. Taubes, *Counting pseudo-holomorphic cirves in dimension 4*, J. Diff. Geom. **44** (1996), 818-893

[Ta3] C. H. Taubes, *SW $\Rightarrow$ Gr: from the Seiberg-Witten equations to pseudoholomorphic curves*, J. Amer. Math. Soc. **9** (1996), 845-918

[Ta4] C. H. Taubes, *Sieberg-Witten and Gromov invariants*, Geometry and Physics (Aarhus, 1995), Lecture notes in pure and applied math., 184, Dekker, New York 1997, 591-601

[Wa] J. Wahl, *Miyaoka-Yau inequality for normal surfaces and local analogues*, Contemporary Mathematics **162** (1994), 381-402

[Wh] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. **36** (1934), 63-89